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Hamiltonian field theory ☆

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Abstract

In this paper, a general Hamiltonian theory for Lagrangian systems on fibred manifolds is proposed. The concept of a *Lepagean* $(n + 1)$*-form* is defined (where n is the dimension of the base manifold), generalizing Krupka's concept of a Lepagean n-form. Lepagean $(n+1)$-forms are used to study Lagrangian and Hamiltonian systems. Innovations and new results concern the following: a *Lagrangian system* is considered as an *equivalence class* of local Lagrangians (of all orders starting from a minimal one); a *Hamiltonian system* is associated with an Euler–Lagrange form (not with a particular Lagrangian); *Hamilton equations* are based upon a Lepagean $(n + 1)$-form, and cover Hamilton–De Donder equations (which are based upon the exterior derivative of the Poincaré–Cartan form) as a special case. First-order Hamiltonian systems, namely those carying higher-degree contact components of the corresponding Lepagean forms, are studied in detail. The presented geometric setting leads to a new (more general than the standard one) understanding of the concepts of *regularity* and *Legendre transformation* in the calculus of variations, relating them directly to the properties of the arising *exterior differential systems*. In this way, new *regularity conditions* and *Legendre transformation formulas* are obtained, depending on a Lepagean $(n+1)$-form, i.e., related with the corresponding *Euler–Lagrange form*.

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1. Introduction

The aim of this paper is to propose a general differential geometric setting for the Hamiltonian field theory in fibred manifolds. Geometric formulations of Hamilton equations in field theory as a part of the calculus of variations on fibred manifolds are connected with the names of many authors (cf. [1,5–8,12,13,16–20,22,23,29,30,35–37,43,47–52,55–58] and references herein).

Our approach is different from the usual one, and leads to a more general setting which *covers the Hamilton–De Donder theory* of the calculus of variations *as a special case*. Moreover, it can be seen as *unifying* and further generalizing different approaches to the Hamilton theory: the "standard" one which goes back to Goldschmidt and Sternberg [20] with several "nonstandard" ones ([5,43], and the very recent by Krupková and Smetanová [49,50]). Main differences and results are the following:

(1) *Lepagean* $(n+1)$*-form*. The key concept in our formulation of Lagrangian and Hamiltonian theories is that of a *Lepagean* $(n+1)$*-form* (where n is the dimension of the base manifold). Lepagean $(n+1)$-forms represent a generalization to $(n+1)$-forms of the fundamental concept of the calculus of variations on fibred manifolds—the *Lepagean n-form*, introduced by Krupka in 1973 [31] (see also [32,35,38,41] for further results). While Krupka's Lepagean n-forms are counterparts of *Lagrangians* (like, e.g., the famous Poincaré–Cartan form which is a particular case of a Lepagean n-form), Lepagean $(n+1)$-forms introduced in this paper are counterparts of *Euler–Lagrange forms* (cf. Lepagean 2-forms in mechanics [44–46]).

(2) *Lagrangian system*. Usually, by a Lagrangian system a *global Lagrangian* is understood. In this paper (similarly to our previous work concerning mechanics [44–46]), the definition is more general, introducing a Lagrangian system as an *equivalence class of Lepagean* $(n+1)$*-forms*. Thus, by a Lagrangian system, we mean the *family of all equivalent Lagrangians* (i.e., Lagrangians whose Euler–Lagrange forms coincide). It should be noted that the equivalence class contains local Lagrangians of all finite orders starting from a certain minimal one; moreover, in general, a global Lagrangian need not exist—the obstructions lie in the topology of the total space of the underlying fibred manifold (see [2,9,39,59–61] and others). A similar approach to Lagrangian systems is applied in [24,26].

(3) *Hamiltonian system*. Contrary to the usual procedure when a Hamiltonian system is associated with a Lagrangian, we define a Hamiltonian system to be a Lepagean $(n+1)$-form. In this way, a Hamiltonian system is associated with an *Euler–Lagrange form* (*not with a particular Lagrangian*), i.e., it is *the same* for all the equivalent Lagrangians. This approach supports the idea that the concept of a Hamiltonian system should reflect only those properties of the corresponding Lagrangians, which are directly related with the *dynamics*. Consequently, the most important physical characteristics of Hamiltonian systems, i.e., *Hamiltonians* and *momenta refer to the whole class of equivalent Lagrangians*.

By definition, to every Hamiltonian system one has a uniquely determined Lagrangian system. On the other hand, since a Lepagean $(n+1)$-form is determined by an Euler–Lagrange form *as well as* by auxiliary (upon the Euler–Lagrange form independent) terms, one has *many* Hamiltonian systems associated with a Lagrangian system.

(4) *Hamilton equations*. First, we adopt the approach of Goldschmidt and Sternberg [20] to understand Hamilton equations as equations for *sections of a prolongation* of the underlying fibred manifold. Within this approach, Hamilton equations are defined intrinsically, and without any a priori assumption on "regularity", or existence of "Legendre transformation". Moreover, Hamilton equations appear as an *extension* of the Euler–Lagrange equations, and regularity and existence of a proper Legendre transformation become an additional property of these equations, which can be specified from geometric requirements.

Next, we develop the idea of Dedecker [5] and Krupka [36] that Hamilton equations related with a Lagrangian could be more generally considered to be based upon a *general* Lepagean equivalent of a Lagrangian, not only upon its Poincaré–Cartan form, as usually done. In our setting, Hamilton equations become equations for integral sections of a *Hamilton exterior differential system arising from a Lepagean* $(n+1)$*-form*, and as such, they become a counterpart of the *Euler–Lagrange equations* (not of a particular Lagrangian). Moreover (similarly as within the approach suggested by Dedecker [5]), they depend not only upon the Lagrangian system itself, but also upon higher-degree contact components of the corresponding Lepagean $(n+1)$-form.

(5) In this paper, *regularity* and *Legendre transformation* for a Hamiltonian system are defined in a geometrical way to be *properties of the corresponding Hamilton exterior differential system*. From such a definition, we derive *regularity conditions* and *Legendre transformation formulas* which depend on the *Euler–Lagrange form* (not on a particular Lagrangian) *and* on the *higher-degree contact terms* in the corresponding Lepagean $(n+1)$-form. Expressing the regularity conditions and the Legendre transformation formulas by means of individual Lagrangians, one gets expressions which may differ from the standard ones. As we show, the presented geometrical concept of regularity brings a unified look at different regularity conditions which have appeared in the literature (the standard one, as well as those in [5,43,49]). On the other hand, our Legendre transformation differs from that proposed by Dedecker in [5]; however, if applied to first-order Lagrangians, it contains both the standard Legendre transformation formulas and those proposed by Krupková and Smetanová [50]. Similarly, for second-order Lagrangians affine in the second derivatives, we recover the formulas discovered by Krupka and Štěpánková [43].

(6) *Strong regularity*. For general first-order Hamiltonian systems regularity is not sufficient to guarantee a *bijective* correspondence between extremals and Hamilton extremals. Therefore, the need to study *equivalence* between the Hamilton and Euler–Lagrange equations leads to the concept of *strong regularity*. We show that for Hamilton–De Donder systems regularity and strong regularity coincide. For general Hamiltonian systems, we find conditions for strong regularity and show relations between strong regularity and existence of Legendre transformations.

(7) *Regularization*. The generalized setting for the Hamilton theory suggests *a new understanding of the role of regularity, Legendre transformation*, and *Hamilton equations in the calculus of variations*. Namely, higher-degree contact terms which appear in the generalized Hamilton equations can be considered as "parameters" giving one the

possibility to search for *appropriate* Hamilton equations (i.e., regular and admitting Legendre transformation) for a given variational problem. From this point of view, we study *regularizations* of some interesting Lagrangians (namely, Lagrangians *affine* or *quadratic in the first derivatives*, and *affine in the second derivatives*). It turns out that these examples cover all physically interesting Lagrangian systems (among them the Dirac field, the scalar field, the electromagnetic field, gravity). It should be mentioned that the possibility of "regularizing" a Lagrangian by means of choosing an appropriate "Lepagean equivalent" has been first noticed by Dedecker in [5].

The plan of the paper is as follows. In Section 2, we recall notations and some preliminary facts on horizontal and contact forms on jet prolongations of fibred manifolds. Section 3 is a review of the Hamilton–De Donder theory. Since, on one hand, this theory is considered to be standard (and, as such, subject of basic monographs—cf., e.g., [18]), and, on the other hand, not quite satisfactory (cf. [7,22,56]), and since there exist a few different approaches leading to "nonstandard" results which are less known but interesting from the point of view of applications in physics (cf. [5,16,28,43]), this section is included as a motivation. The core of the paper is Section 4 where our setting is explained, new results are stated, and links to known results are mentioned. We concentrate ourself to *first-order Hamiltonian systems* (which in this approach concern also some *second-order Lagrangians*). The theory can be generalized to the higher-order in a quite straightforward way [47,48].

2. Notations and preliminaries

Throughout the paper, $\pi : Y \to X$ is a *fibred manifold* with a base X, $\dim X = n$, and a total space Y, $\dim Y = n + m$. For every $x \in X$, the submanifold $\pi^{-1}(x) \in Y$ is called a *fibre* over x. We denote by (V, ψ), $\psi = (x^i, y^\sigma)$ a fibred chart on Y. A (smooth) mapping $\gamma : U \to Y$, where U is an open subset of X is called a *section* of the fibred manifold π if $\pi \circ \gamma = \mathrm{id}_U$. For $s \geq 1$, the *s-jet prolongation* of a fibred manifold π is denoted by $\pi_s : J^sY \to X$. The s-jet prolongation of a section γ of π is denoted by $J^s\gamma$; it is a section of π_s. A section δ of π_s is called *holonomic* if there exists a section γ of π such that $\delta = J^s\gamma$. To every fibred chart (V, ψ), $\psi = (x^i, y^\sigma)$ on π there exists the so-called *associated chart* on J^sY, denoted by (V_s, ψ_s), $\psi_s = (x^i, y^\sigma, y^\sigma_{j_1\cdots j_k})$, where $V_s = \pi^{-1}_{s,0}(V)$, $1 \leq \sigma \leq m$, and $1 \leq k \leq s$, $1 \leq j_1 \leq j_2 \leq \cdots \leq j_k \leq n$. By $\pi_{s,k}$, where $0 \leq k < s$, we denote the natural fibred projections, $\pi_{s,k} : J^sY \to J^kY$.

A vector field ξ on J^sY is called *π_s-vertical* if $T\pi_s \cdot \xi = 0$. The bundle of π_s-vertical vectors is obviously a subbundle of the tangent bundle TJ^sY; it will be denoted by $V\pi_s$. In an analogous way, one defines the concept of a *$\pi_{s,k}$-vertical* vector field on J^sY for $0 \leq k < s$. The *s-jet prolongation* of a π-vertical vector field ξ on Y is denoted by $J^s\xi$; it is a vector field on J^sY.

Denote by $\Lambda^q(J^sY)$ the module of q-forms on J^sY over the ring of functions. A form $\eta \in \Lambda^q(J^sY)$ is called *$\pi_{s,k}$-projectable* if there exists a form $\eta_0 \in \Lambda^q(J^kY)$ such that $\pi^*_{s,k}\eta_0 = \eta$; the form η_0 is then called the *$\pi_{s,k}$-projection* of η. A form $\eta \in \Lambda^q(J^sY)$ is called *π_s-horizontal* if $i_\xi\eta = 0$ for every π_s-vertical vector field ξ on J^sY. Similarly, a form $\eta \in \Lambda^q(J^sY)$ is called *$\pi_{s,k}$-horizontal*, $0 \leq k < s$, if $i_\xi\eta = 0$ for every $\pi_{s,k}$-vertical vector field ξ on J^sY. The module of π_s-horizontal (respectively, $\pi_{s,k}$-horizontal) q-forms on J^sY is a submodule of $\Lambda^q(J^sY)$ and is denoted by $\Lambda^q_X(J^sY)$ (respectively, $\Lambda^q_{J^kY}(J^sY)$). We denote by h the *horizontalization* of differential forms. h is an R-linear, wedge product preserving mapping, assigning to $\eta \in \Lambda^q(J^sY)$ a form $h\eta \in \Lambda^q(J^{s+1}_XY)$, and is defined by the formulas

$$h\,\mathrm{d}x^i = \mathrm{d}x^i, \qquad h\,\mathrm{d}y^\sigma_{j_1\cdots j_k} = y^\sigma_{j_1\cdots j_k i}\,\mathrm{d}x^i, \quad 0 \le k \le s, \quad hf = f \circ \pi_{s+1,s}.$$

We can see that

$$h\,\mathrm{d}f = d_i f\,\mathrm{d}x^i, \quad d_i f = \frac{\partial f}{\partial x^i} + \sum_{k=0}^{s} \frac{\partial f}{\partial y^\sigma_{j_1\cdots j_k}} y^\sigma_{j_1\cdots j_k i}.$$

Apparently, for $q > \dim X$, $h\eta = 0$. A form $\eta \in \Lambda^q(J^sY)$, $q \ge 0$, is called *contact* if $J^s\gamma^*\eta = 0$ for every section γ of π. Obviously, η is contact if and only if $h\eta = 0$. For $s \ge 1$ denote

$$\omega^\sigma_{j_1\cdots j_k} = \mathrm{d}y^\sigma_{j_1\cdots j_k} - y^\sigma_{j_1\cdots j_k i}\,\mathrm{d}x^i, \tag{2.1}$$

where $1 \le \sigma \le m$, $0 \le k \le s-1$, $j_1, \dots, j_k = 1, 2, \dots n$. The above (local) 1-forms are contact forms on J^sY. It is worthwhile to note that

$$(\mathrm{d}x^i, \omega^\sigma_{j_1\cdots j_k}, \mathrm{d}y^\sigma_{j_1\cdots j_s}), \quad 0 \le k \le s-1, \quad 1 \le j_1 \le \cdots \le j_s \le n \tag{2.2}$$

is a *basis of linear forms* on $V_s \subset J^sY$. Note that

$$d\omega^\sigma_{j_1\cdots j_k} = -\mathrm{d}y^\sigma_{j_1\cdots j_k i} \wedge \mathrm{d}x^i = -\omega^\sigma_{j_1\cdots j_k i} \wedge \mathrm{d}x^i.$$

According to [40], the ideal of contact forms on J^sY, called the *contact ideal*, is locally generated by the forms $\omega^\sigma, \omega^\sigma_{j_1}, \dots, \omega^\sigma_{j_1\cdots j_{s-1}}, \mathrm{d}\omega^\sigma_{j_1\cdots j_{s-1}}$. The contact ideal plays an important role in the calculus of variations and the theory of differential equations on manifolds, since it enables one to "recognize" holonomic sections: *A section δ of a fibred manifold π_s is holonomic if and only if it is an integral section of the contact ideal on J^sY.*

The definition of a contact form implies that every q-form η on J^sY, where $q > n$ is contact. Let us turn to a "softer" classification of contact forms, suggested by the fibred structure. Let $q \ge 1$, and let $\eta \in \Lambda^q_{J^{s-1}Y}(J^sY)$ be a *contact* form. We say that η is 1-*contact* if for each π_s-vertical vector field ξ on J^sY the $(q-1)$-form $i_\xi\eta$ is π_s-horizontal; we say that η is k-*contact*, $2 \le k \le q$, if $i_\xi\rho$ is $(k-1)$-contact [35]. In this context, horizontal forms are also called 0-*contact*. Hence, η is i-contact if and only if each term in its coordinate expression with respect to a basis (2.2) contains *exactly* i of the 1-contact linear forms (2.1). The following is a basic theorem on the structure of forms on fibred manifolds.

Decomposition theorem (Krupka [35]). *Every $\eta \in \Lambda^q_{J^{s-1}Y}(J^sY)$ is uniquely decomposable in the form $\eta = \eta_0 + \eta_1 + \cdots + \eta_q$, where η_i, $0 \le i \le q$, is a i-contact form on J^sY.*

If η_i is the i-contact part of η, we write $\eta_i = p_i\eta$. In this way, for *every* q-form η on J^sY, we obtain a unique invariant decomposition

$$\pi^*_{s+1,s}\eta = h\eta + p_1\eta + \cdots + p_q\eta \tag{2.3}$$

into a sum of a horizontal form and i-contact q-forms, $1 \le i \le q$. This formula is fundamental for our computations and will be frequently used throughout the paper.

In what follows, we shall use the following notations:

$$\omega_0 = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n, \qquad \omega_i = i_{\partial/\partial x^i}\omega_0,$$
$$\omega_{i_1\cdots i_l} = i_{\partial/\partial x^{i_l}}\omega_{i_1\cdots i_{l-1}}, \quad 2 \le l \le n. \tag{2.4}$$

Note that $\mathrm{d}x^i \wedge \omega_j = \delta^i_j\omega_0$, $\mathrm{d}x^i \wedge \omega_{jk} = \delta^i_k\omega_j - \delta^i_j\omega_k$, etc.

3. A brief review of Hamilton–De Donder theory

3.1. First-order Lagrangians

Inspired by the work of De Donder [10], Golschmidt and Sternberg [20] in their famous paper set geometric foundations of a Hamilton theory on fibred manifolds, which is known as the *Hamilton–De Donder theory*. Main ideas can be very briefly summarized as follows.

Consider a fibred manifold $\pi : Y \to X$, $\dim X = n$, $\dim Y = m + n$, and its first jet prolongation $\pi_1 : J^1Y \to X$. Let λ be a first-order Lagrangian, i.e., a horizontal n-form on J^1Y, and θ_λ the *Poincaré–Cartan form* of λ [14,20,31]. In a fibred chart (V, ψ), $\psi = (x^i, y^\sigma)$ on Y one gets $\lambda = L\omega_0$, where L is a function on $\pi^{-1}_{1,0}(V)$, and

$$\theta_\lambda = L\omega_0 + \frac{\partial L}{\partial y^\sigma_j}\,\omega^\sigma \wedge \omega_j. \tag{3.1}$$

A section γ of π, defined on an open subset $U \in X$, is called an *extremal* of the Lagrangian λ (over a compact n-dimensional submanifold $\Omega \subset X$ with boundary $\partial\Omega$) if for every π-vertical vector field ξ on Y,

$$\int_\Omega J^1\gamma^*\partial_{J^1\xi}\lambda = 0. \tag{3.2}$$

By a direct computation, one gets that $\gamma : U \to Y$ is an extremal of λ if and only if

$$J^1\gamma^* i_{J^1\xi}\,\mathrm{d}\theta_\lambda = 0 \quad \text{for every } \pi\text{-vertical vector field } \xi \text{ on } Y. \tag{3.3}$$

Eq. (3.3) is an intrinsic version of the *Euler–Lagrange equations* of λ; in fibred coordinates it takes the familiar form of m second-order PDE for the components of γ,

$$\left(\frac{\partial L}{\partial y^\sigma} - d_j\frac{\partial L}{\partial y^\sigma_j}\right) \circ J^2\gamma = 0. \tag{3.4}$$

Goldschmidt–Sternberg's geometric approach to Hamilton equations is based on the idea to understand them as equations *for sections of the prolonged manifold* $J^1Y \to X$. Namely, a section $\delta : U \to J^1Y$ of the fibred manifold π_1 is called a *Hamilton extremal* of the Lagrangian λ if

$$\delta^* i_\xi d\theta_\lambda = 0 \quad \text{for every } \pi_1\text{-vertical vector field } \xi \text{ on } J^1Y. \tag{3.5}$$

Eq. (3.5) represent $2mn$ first-order PDE for the components $(\delta^\sigma, \delta_i^\sigma)$ of sections of π_1; they are called *Hamilton–De Donder equations*. One can easily see that *if γ is an extremal of λ then $J^1\gamma$ is its Hamilton extremal*. On the other hand, *a Hamilton extremal generally need not be of the form of a prolongation of an extremal*. In this sense, the Hamilton theory is an *extension* of the Lagrange theory.

Theorem 3.1 (Goldschmidt and Sternberg [20]). *If λ satisfies the condition*

$$\det\left(\frac{\partial^2 L}{\partial y_j^\sigma \partial y_k^\nu}\right) \neq 0 \tag{3.6}$$

at each point of J^1Y, then every Hamilton extremal of λ is of the form $\delta = J^1\gamma$, where γ is an extremal of λ.

Consequently, provided the *regularity condition* (3.6) is satisfied, the sets of extremals and of Hamilton extremals of λ are in *bijective* correspondence, i.e., the Hamilton–De Donder equations are *equivalent* with the Euler–Lagrange equations, and this bijection is realized via the J^1 prolongation mapping.

The concepts of Hamiltonian, momenta and Legendre transformation are obtained with help of the Poincaré–Cartan form θ_λ. Namely, we can write

$$\theta_\lambda = \left(L - \frac{\partial L}{\partial y_j^\sigma} y_j^\sigma\right)\omega_0 + \frac{\partial L}{\partial y_j^\sigma}\,\mathrm{d}y^\sigma \wedge \omega_j, \tag{3.7}$$

and put

$$H = -L + \frac{\partial L}{\partial y_j^\sigma} y_j^\sigma, \qquad p_\sigma^j = \frac{\partial L}{\partial y_j^\sigma}. \tag{3.8}$$

In analogy with mechanics, the functions H and p_σ^j, $1 \le j \le n$, $1 \le \sigma \le m$, are called the *Hamiltonian* and *momenta* of λ. Obviously, *if the regularity condition* (3.6) *is satisfied then*

$$(x^i, y^\sigma, y_j^\sigma) \to (x^i, y^\sigma, p_\sigma^j) \tag{3.9}$$

is a local coordinate transformation on J^1Y; it is called Legendre transformation. Writing the Hamilton–De Donder equations (3.5) in the Legendre coordinates one gets them in the familiar form

$$\frac{\partial y^\sigma}{\partial x^j} = \frac{\partial H}{\partial p_\sigma^j}, \qquad \frac{\partial p_\sigma^j}{\partial x^j} = -\frac{\partial H}{\partial y^\sigma}. \tag{3.10}$$

We remark that, contrary to (3.1), the decomposition (3.7) of θ_λ into a sum of two terms is noninvariant with respect to fibred transformations. This means, in particular, that the Hamiltonian H (respectively, the n-form $H\omega_0$) is defined only locally. However, a concept of a "global Hamiltonian" can be obtained easily; such an n-form is called an *extended Lagrangian*, and its role in the Hamiltonian setting is analogous to that of a Lagrangian in the Lagrange theory (for more details see [43], for higher-order [37]).

3.2. Higher-order Lagrangians

The above mentioned Goldschmidt–Sternberg's setting for Hamilton theory in fibred manifolds has been generalized to the case of Lagrangians of an arbitrary order r during the period 1980–1990 [1,8,12,15,17,22,23,30,35–37,52,57].

Consider a Lagrangian of order r, i.e., a horizontal n-form λ on J^rY. In a fibred chart,

$$\lambda = L\omega_0, \quad L = L(x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \dots, y^\sigma_{j_1 j_2 \cdots j_r}). \tag{3.11}$$

As pointed out already by De Donder in 1930 [10], one can assign to L an n-form

$$\theta_\lambda = L\omega_0 + \sum_{k=0}^{r-1} \left(\sum_{l=0}^{r-k-1} (-1)^l d_{p_1} d_{p_2} \cdots d_{p_l} \frac{\partial L}{\partial y^\sigma_{j_1 \dots j_k p_1 \cdots p_l i}} \right) \omega^\sigma_{j_1 \cdots j_k} \wedge \omega_i, \tag{3.12}$$

defined on $V_{2r-1} = \pi^{-1}_{2r-1,0}(V) \subset J^{2r-1}Y$; it is called the *(higher-order) Poincaré–Cartan equivalent of* λ. Based upon this form, Shadwick obtained in [57] a direct generalization of the first-order Hamilton–De Donder theory, now called *local Hamilton–De Donder theory*. It can be summarized as follows: a section δ of the fibred manifold π_{2r-1}, passing in V_{2r-1}, is called a *Hamilton extremal* of λ if

$$\delta^* i_\xi \, \mathrm{d}\theta_\lambda = 0 \quad \text{for every } \pi_{2r-1}\text{-vertical vector field } \xi \text{ on } V_{2r-1}. \tag{3.13}$$

Eq. (3.13) (which are first-order PDE) are called *Hamilton–De Donder equations*. The point is to study their relation with the Euler–Lagrange equations which are PDE of order $2r$ for sections γ of π,

$$J^{2r-1}\gamma^* i_{J^{2r-1}\xi} d\theta_\lambda = 0 \quad \text{for every } \pi\text{-vertical vector field } \xi \text{ on } Y. \tag{3.14}$$

For the purpose of the next theorem, let us denote by $[q_1 \cdots q_s]$ the *number* of all different sequences arising by permuting the sequence $q_1, \dots, q_s$. It holds

$$[q_1 \cdots q_s] = \frac{s!}{i_1! \cdots i_n!},$$

where i_k is the number of the integers k in the sequence $q_1, \dots, q_s$.

Theorem 3.2 (Shadwick [57]). *Let* λ *be a Lagrangian on* J^rY, $V \subset Y$ *a fibred chart. On* V_{2r-1} *consider the Poincaré–Cartan equivalent* θ_λ (3.12) *of* λ, *rewritten in the form*

$$\theta_\lambda = -H\omega_0 + p^i_\sigma \, \mathrm{d}y^\sigma \wedge \omega_i + p^{j_1 i}_\sigma \, \mathrm{d}y^\sigma_{j_1} \wedge \omega_i + \cdots + p^{j_1 \cdots j_{r-1} i}_\sigma \, \mathrm{d}y^\sigma_{j_1 \cdots j_{r-1}} \wedge \omega_i, \tag{3.15}$$

where

$$p^{j_1 \cdots j_k i}_\sigma = \sum_{l=0}^{r-k-1} (-1)^l \, d_{p_1} d_{p_2} \cdots d_{p_l} \frac{\partial L}{\partial y^\sigma_{j_1 \cdots j_k p_1 \cdots p_l i}}, \quad 0 \le k \le r-1,$$

$$H = -L + \sum_{k=1}^{r} p^{j_1 \cdots j_k}_\sigma y^\sigma_{j_1 \cdots j_k}. \tag{3.16}$$

Consider the family of matrices

$$\left(\frac{1}{[j_1\cdots j_{2r-s}(p_{r+1}\cdots p_s][p_1\cdots p_r)]}\frac{\partial^2 L}{\partial y^\sigma_{j_1\cdots j_{2r-s}(p_{r+1}\cdots p_s}\partial y^\nu_{p_1\cdots p_r)}}\right), \tag{3.17}$$

where $r\le s\le 2r-1$, *the* σ, $j_1\le\cdots\le j_{2r-s}$ *label columns and* ν, $p_1\le\cdots\le p_s$ *label rows, and the brackets* $(\cdots)$ *denote symmetrization in the indicated indices. If ranks of all of these matrices are maximal then the system of functions*

$$x^i,\ y^\sigma,\ y^\sigma_{j_1},\ldots,y^\sigma_{j_1\cdots j_{r-1}},\ p^{j_1\cdots j_r}_\sigma,\ p^{j_1}_\sigma,\quad j_1\le\cdots\le j_r \tag{3.18}$$

forms a part of a coordinate system on W, and every Hamilton extremal δ *passing in* V_{2r-1} *is of the form* $\pi_{2r-1,r}\circ\delta=J^r\gamma$, *where* γ *is an extremal of* λ.

Shadwick's regularity condition has been formally geometrized in [17,30]; in these papers a geometric version of the matrices (3.17) by means of bilinear forms (respectively, equivalently, by a linear mapping) was given.

The functions H and p's are called a *Hamiltonian* and *momenta* of the Lagrangian λ, and coordinates based on (3.18) are called *Legendre coordinates*. In any such coordinates Hamilton–De Donder equations (3.13) take the following "canonical" form (cf. [8,10,22,37,57]):

$$\frac{\partial y^\sigma_{j_1\cdots j_k}}{\partial x^i}=\frac{\partial H}{\partial p^{j_1\cdots j_k i}_\sigma},\qquad \frac{\partial p^{j_1\cdots j_k l}_\sigma}{\partial x^l}=-\frac{\partial H}{\partial y^\sigma_{j_1\cdots j_k}}, \tag{3.19}$$

where $0\le k\le r-1$, and in the second set of equations, summation over l takes place.

3.3. *Problems*

The above mentioned approach to Hamilton theory is considered more or less standard within the calculus of variations. However, unfortunately, it suffers from many inconveniences and problems both from the point of view of mathematics and physics. Let us mention some of the most serious ones.

3.3.1. *Effects of nonuniqueness in higher-order*

Geometric studies of Poincaré–Cartan equivalents for Lagrangians of order $r\ge 2$ in field theory resulted in a striking result: formula (3.12) for θ_λ generally does *not* give rise to a *globally* defined form on $J^{2r-1}Y$. A "globalization" is possible, however, is paid by *nonuniqueness*. There appeared a lot of papers dealing with this problem and providing different constructions of global higher-order Poincaré–Cartan forms; we refer, e.g., to [8,11,12,15,21,27,29,35]. It should be stressed that, on the other hand, within a global variational theory on fibred manifolds based upon the concept of a *Lepagean n-form* ($n=$ the dimension of the base manifold X), developed by Krupka since 1971, "true" (global) Poincaré–Cartan forms appear naturally as special cases of more general *Lepagean equivalents of a Lagrangian* [31,32,38,41], cf. also [21,51,53]. Also, the role of the local form θ_λ is clarified. Let us recall some of the results on Lepagean n-forms we shall need later.

Let $s \geq 0$. An n-form ρ on J^sY is called a *Lepagean n-form* (*of order s*) if $hi_\xi \,\mathrm{d}\rho = 0$ for every $\pi_{s+1,0}$-vertical vector field ξ on $J^{s+1}Y$. The horizontal part $h\rho$ of ρ is an n-form on $J^{s+1}Y$, i.e., a *Lagrangian* of order $s+1$. We denote

$$\lambda = h\rho, \qquad \lambda = L\omega_0, \tag{3.20}$$

and say that ρ is a *Lepagean equivalent* of the Lagrangian λ. Conversely, it can be shown [35,41,54] that every Lagrangian has a (global) Lepagean equivalent.

Theorem 3.3 (Krupka [35,41]). *The following conditions are equivalent:*

(1) *ρ is a Lepagean n-form of order s.*
(2) *The $(n+1)$-form $p_1\,\mathrm{d}\rho$ is $\pi_{s+1,0}$-horizontal.*
(3) *In every fibred chart (V,ψ), $\psi = (x^i, y^\sigma)$, on Y,*

$$\pi^*_{s+1,s}\rho = \theta_\lambda + \mathrm{d}\nu + \mu, \tag{3.21}$$

where ν is a contact $(n-1)$-form, and μ is an n-form which is at least 2-contact.

The splitting (3.21) is, in general, not coordinate independent. Therefore, in higher-order field theory, for a global Lagrangian one generally has not a global associated Poincaré–Cartan equivalent θ_λ. On the other hand, the forms

$$\Theta = \lambda + p_1\rho = \theta_\lambda + p_1\,\mathrm{d}\nu \tag{3.22}$$

represent *all global Lepagean equivalents of λ which are at most 1-contact*.

If ρ is a Lepagean n-form then

$$\pi^*_{s+1,s}\,\mathrm{d}\rho = E_\lambda + F, \tag{3.23}$$

where E_λ is a 1-contact $\pi_{s+1,0}$-horizontal $(n+1)$-form, and F is an $(n+1)$-form which is at least 2-contact. E_λ is called the *Euler–Lagrange form* of the Lagrangian λ; in every fibred chart,

$$E_\lambda = \left(\frac{\partial L}{\partial y^\sigma} - \sum_{l=1}^{s+1}(-1)^l d_{p_1}d_{p_2}\cdots d_{p_l}\frac{\partial L}{\partial y^\sigma_{p_1p_2\cdots p_l}}\right)\omega^\sigma \wedge \omega_0, \tag{3.24}$$

i.e., components of E_λ are the *Euler–Lagrange expressions*. Apparently, if λ is defined on J^rY then its Euler–Lagrange form is of order $\leq 2r$. Comparing (3.23) with (3.22) and (3.21) one can see that

$$p_1\,\mathrm{d}\theta_\lambda = p_1\,\mathrm{d}\rho = p_1\,\mathrm{d}\Theta = E_\lambda, \tag{3.25}$$

i.e., contrary to θ_λ, the form $p_1\,\mathrm{d}\theta_\lambda$ is defined *globally*, and E_λ is *uniquely* determined by the Lagrangian λ (not depending upon ν).

Going back to the Hamilton–De Donder theory, we have for a higher-order Lagrangian, on the Lagrangian side, due to (3.25), *unique*, *global* Euler–Lagrange equations

$$J^{2r-1}\gamma^* i_{J^{2r-1}\xi}\,\mathrm{d}\Theta = 0 \quad \text{for every } \pi\text{-vertical vector field } \xi \text{ on } Y, \tag{3.26}$$

and, on the Hamiltonian side, either *many local* Hamilton–De Donder equations, among which there are the following equations *uniquely* determined by the Lagrangian,

$$\delta^* i_\xi \, \mathrm{d}\theta_\lambda = 0 \quad \text{for every } \pi_{2r-1}\text{-vertical vector field } \xi, \tag{3.27}$$

or *global* Hamilton–De Donder equations of the form

$$\delta^* i_\xi \, \mathrm{d}\Theta = 0 \quad \text{for every } \pi_{2r-1}\text{-vertical vector field } \xi \text{ on } J^{2r-1}Y, \tag{3.28}$$

which, however, *depend upon an auxiliary term* $p_1 \, \mathrm{d}\nu$.

Consequently, one could expect that also the regularity condition and the Legendre transformation should depend upon $p_1 \, \mathrm{d}\nu$. There is, however, the following interesting and rather striking result due to Krupka [36] (later obtained also by Gotay [22]), saying that *regularity does not depend upon the term* $p_1 \, \mathrm{d}\nu$. More precisely,

Theorem 3.4 (Krupka [36]). *Let λ be a Lagrangian of order r. Consider its Lepagean equivalent Θ (3.22) on V_{2r-1}, and set*

$$\Theta = -\bar{H}\omega_0 + \bar{p}^i_\sigma \, \mathrm{d}y^\sigma \wedge \omega_i + \bar{p}^{j_1 i}_\sigma \, \mathrm{d}y^\sigma_{j_1} \wedge \omega_i + \cdots + \bar{p}^{j_1\cdots j_{r-1} i}_\sigma \, \mathrm{d}y^\sigma_{j_1\cdots j_{r-1}} \wedge \omega_i + \sum_{k=r}^{2r-2} q^{j_1\cdots j_k i}_\sigma \, \mathrm{d}y^\sigma_{j_1\cdots j_k} \wedge \omega_i, \tag{3.29}$$

where

$$\bar{p}^{j_1\cdots j_k i}_\sigma = \sum_{l=0}^{r-k-1} (-1)^l d_{p_1} d_{p_2} \cdots d_{p_l} \frac{\partial L}{\partial y^\sigma_{j_1\cdots j_k p_1 \cdots p_l i}} + q^{j_1\cdots j_k i}_\sigma, \quad 0 \le k \le r-1,$$

$$\bar{H} = -L + \sum_{k=1}^{r} \bar{p}^{j_1\cdots j_k}_\sigma y^\sigma_{j_1\cdots j_k} + \sum_{k=r+1}^{2r-1} q^{j_1\cdots j_k}_\sigma y^\sigma_{j_1\cdots j_k}. \tag{3.30}$$

Let $x \in V_{2r-1}$ be a point. If the Shadwick regularity conditions in a neighbourhood W of x are satisfied then the system of functions

$$x^i, y^\sigma, y^\sigma_{j_1}, \ldots, y^\sigma_{j_1\cdots j_{r-1}}, \bar{p}^{j_1\cdots j_r}_\sigma, \ldots, \bar{p}^{j_1}_\sigma, \quad j_1 \le \cdots \le j_r \tag{3.31}$$

is a part of a coordinate system on W, and every Hamilton extremal δ passing in W is of the form $\pi_{2r-1,r} \circ \delta = J^r\gamma$, where γ is an extremal of λ.

Any local coordinates on $J^{2r-1}Y$ based (3.31), i.e., $(x^i, y^\sigma, y^\sigma_{j_1}, \ldots, y^\sigma_{j_1\cdots j_{r-1}}, \bar{p}^{j_1\cdots j_r}_\sigma, \ldots, \bar{p}^{j_1}_\sigma, z^J)$, where z^J's are arbitrary, are called *Legendre coordinates* of the Lagrangian λ. Unfortunately, in general, *Legendre coordinates do not provide Hamilton–De Donder equations in a "canonical form"*, since the form $p_1 \, \mathrm{d}\nu$ (i.e., the functions $q^{j_1\cdots j_k i}_\sigma$ in (3.29) and (3.30)) may depend upon the additional coordinate functions z^J completing (3.31) to a chart.

The fact that for higher-order Lagrangians there arise many different possibilities for Hamilton–De Donder equations which are not completely determined by the Lagrangian, as well as "bad" properties of the above higher-order Legendre transformations (namely, that one has not enough momenta to create a new chart, and that, generally, Legendre coordinates are not useful for obtaining Hamilton equations in a "canonical form"), have been considered very strange and unsatisfactory (cf. [5,7,17,22]). They even led Dedecker to express his doubts about Hamilton theory for higher-order Lagrangians. In his opinion, it is impossible to create a satisfactory Hamiltonian counterpart of Lagrange theory *unless the meaning of Hamilton equations, regularity, and Legendre transformation is properly understood* [7].

3.3.2. Equivalent Lagrangians

Another unsatisfactory point concerns *the role of equivalent Lagrangians* in Hamiltonian field theory [56]. Recall that two Lagrangians λ_1 and λ_2 are called *equivalent* if their Euler–Lagrange forms coincide, i.e., if (possibly up to a projection) $E_{\lambda_1} = E_{\lambda_2}$. It is known, however, that equivalent field Lagrangians (even of the same order), can differ with respect to the property of *regularity*. To illustrate this explicitly, consider the following example [56]. Take the fibred manifold $Y = R^2 \times R^2$ over $X = R^2$ with canonical coordinates denoted by (x, y, u, v) and (x, y) on Y and X, respectively. On $J^1(R^2 \times R^2)$ consider the Lagrange functions

$$L_1 = u_x^2, \qquad L_2 = u_x^2 + u_x v_y - u_y v_x. \tag{3.32}$$

It is easy to see that they are equivalent. However, checking the regularity condition (3.6), we get that L_2 is regular while L_1 is not. This means that the Hamilton–De Donder equations $\delta^* i_\xi \, d\theta_{\lambda_1} = 0$ are equivalent with the Euler–Lagrange equations (in other words, can be alternatively used to solve the extremal problem), while the Hamilton–De Donder equations $\delta^* i_\xi \, d\theta_{\lambda_2} = 0$ are "constrained" and cannot be used to solve the original extremal problem in a straightforward way. Thus, although the Lagrangians in (3.32) are equivalent their Hamilton equations are *essentially* different.

The above example suggests an idea that one should better *associate Hamilton equations with Euler–Lagrange equations than with a particular Lagrangian*. Moreover, if Hamilton equations are understood as *alternative* equations describing an *extremal problem*, the key point should be to *choose* in the family of all associated Hamilton equations certain *most appropriate* ones. When applied within higher-order mechanics, this approach led to a generalized setting for the Hamilton theory, with a new understanding of regularity and Legendre transformation, and their role in the theory of variational equations [44–46]. A generalization of that ideas and results to field theory is subject of Section 4.

3.3.3. Almost no "true" applications

In our opinion, the most serious point which makes the standard Hamiltonian field theory unsatisfactory is the fact that it has almost no direct applications in physics: indeed, almost all important physical fields are *singular* (e.g. the Dirac field, the electromagnetic field, the Yang–Mills field, gravity). This means that to study physical fields, additional techniques have to be developed and applied, namely the *Dirac theory of constraints*. Unfortunately, this brings new complications and troubles (cf. [19]). On the other hand, in what follows we shall see that within a new setting, the above mentioned "singular" Lagrangians turn to be no more singular, and there is *no need to apply constraint techniques* for obtaining Hamilton equations for them.

3.4. Fresh ideas

Independently of the above mentioned “standard” Hamilton–De Donder theory there appeared some other ideas bringing a new insight into the problem of developing a Hamiltonian counterpart to the Euler–Lagrange equations.

3.4.1. Dedecker’s Hamilton equations

In 1977, Dedecker [5] published a paper containing a completely different Hamiltonian formulation of the first-order field theory. His point was to find Hamilton equations for *first-order Lagrangians* defined on *contact elements*. Since in this situation there is no natural fibred structure, and the forms $\lambda = L\omega_0$ are not invariant, he took in place of a Lagrangian λ the n-form

$$\rho = L\omega_0 + \frac{\partial L}{\partial y^\sigma_j}\omega^\sigma \wedge \omega_j + \sum \Lambda^{j_1 j_2}_{\sigma_1\sigma_2}\omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge \omega_{j_1 j_2} + \cdots + \sum \Lambda^{j_1\cdots j_n}_{\sigma_1\cdots\sigma_n}\omega^{\sigma_1} \wedge \cdots \wedge \omega^{\sigma_n} \wedge \omega_{j_1\cdots j_n}, \tag{3.33}$$

where the $\Lambda^{j_1 j_2}_{\sigma_1\sigma_2}, \ldots, \Lambda^{j_1\cdots j_n}_{\sigma_1\cdots\sigma_n}$ are *arbitrary* functions of the coordinates (x^k, y^ρ, y^ρ_p), and the indicated summation extends only over increasing sequences of indices. Reformulating and summarizing Dedecker’s results for the *fibred* case, we get the following theorem.

Theorem 3.5 (Dedecker [5]). *Consider the equation*

$$\delta^* i_\xi \rho = 0 \quad \text{for every } \pi_1\text{-vertical vector field } \xi \text{ on } J^1Y. \tag{3.34}$$

Suppose that the condition

$$\det\left(\frac{\partial^2 L}{\partial y^\sigma_j \partial y^\nu_k} - \Lambda^{ij}_{\sigma\nu}\right) \neq 0 \tag{3.35}$$

is satisfied. Then, every solution δ of (3.34) which is an integral section of the ideal generated by the n-forms

$$\omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge \omega_{i_1 i_2}, \qquad \omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge \omega^{\sigma_3} \wedge \omega_{i_1 i_2 i_3}, \qquad \ldots \qquad \omega^{\sigma_1} \wedge \cdots \wedge \omega^{\sigma_n}, \tag{3.36}$$

is holonomic (i.e., $\delta = J^1\gamma$), and its projection $\gamma = \pi_{1,0} \circ \delta$ is an extremal of the Lagrangian L.

We shall call Eq. (3.34) for sections annihilating (3.36) *Dedecker–Hamilton equations*, and the condition (3.35) *Dedecker regularity condition*. In view of results mentioned above, Dedecker–Hamilton equations represent a generalization of the Hamilton–De Donder equations to the case when, in place of the Poincaré–Cartan form θ_λ, a general Lepagean equivalent of a first-order Lagrangian is considered. The regularity condition (3.35) ensures a *bijective correspondence* between *extremals* and a *subset of Hamilton extremals*—the family of integral sections of (3.36).

As pointed out by Dedecker, the dependence of his regularity condition upon the “parameters” $\Lambda^{ij}_{\sigma\nu}$ brings a new possibility for understanding the role of regularity in the calculus of variations. Namely, for a given Lagrangian L one can pose a question whether there *exist* functions $\Lambda^{ij}_{\sigma\nu}$ such that the condition (3.35) is satisfied. Dedecker also illustrated this "regularization procedure" explicitly, showing that it works in the case of a two dimensional electromagnetic field [5].

3.4.2. Krupka–Štěpánková's regularity

Another approach to Hamilton theory, based on Hamilton–De Donder equations, has been proposed by Krupka and Štěpánková [43]. The idea was that the "true order" of the Hamilton–De Donder equations must be taken into account. More precisely, for some Lagrangians of order $r \geq 2$, their Poincaré–Cartan form is $\pi_{2r-1,s}$*-projectable*, where $s < 2r-1$; in this case, it is apparently inappropriate to apply the standard procedure leading to considering Hamilton equations of order $2r-1$. *The problem must be better studied as a problem of order s.*

In [43], Krupka and Štěpánková applied their idea to an important class of *second-order Lagrangians* with $\pi_{2,1}$*-projectable* θ_λ. Let us mention some of the main results. The first point is the *definition of regularity*, different from (3.6): *a Lagrangian is called regular if every its Hamilton extremal is holonomic*. Next, the following second-order Lagrangians affine in the second derivatives were considered: $\lambda = L\omega_0$, where

$$L = L_0(x^i, y^\sigma, y^\sigma_j) + h^{pq}_\nu(x^i, y^\sigma) y^\nu_{pq}. \tag{3.37}$$

It can be seen that the expression of L in the form (3.37) is saved with respect to fibred transformations, and the Poincaré–Cartan form θ_λ is *projectable* onto J^1Y. Thus, the dynamical space for such extremal problems is J^1Y (and not J^3Y, as usually considered), and the Euler–Lagrange equations, respectively, the Hamilton–De Donder equations read as follows:

$$\begin{aligned} &J^1\gamma^* i_{J^1\xi}\, \mathrm{d}\theta_\lambda = 0 \quad \text{for every } \pi\text{-vertical vector field } \xi \text{ on } Y, \\ &\delta^* i_\xi\, \mathrm{d}\theta_\lambda = 0 \quad \text{for every } \pi_1\text{-vertical vector field } \xi \text{ on } J^1Y. \end{aligned} \tag{3.38}$$

Although in the sense of the "conventional" regularity condition (3.17), Lagrangians (3.37) are apparently *singular*, the following theorem holds.

Theorem 3.6 (Krupka and Štěpánková [43]). *Let λ be a Lagrangian of the form (3.37). If the condition*

$$\det\left(\frac{\partial^2 L_0}{\partial y^\sigma_i \partial y^\nu_k} - \frac{\partial h^{ik}_\sigma}{\partial y^\nu} - \frac{\partial h^{ki}_\nu}{\partial y^\sigma}\right) \neq 0 \tag{3.39}$$

is satisfied then λ is regular, the Euler–Lagrange and the Hamilton–De Donder equations (3.38) are equivalent, and the mapping

$$(x^i, y^\sigma, y^\sigma_j) \to (x^i, y^\sigma, p^j_\sigma), \quad p^j_\sigma = \frac{\partial L_0}{\partial y^\sigma_j} - \frac{\partial h^{jk}_\sigma}{\partial x^k} - \left(\frac{\partial h^{jk}_\sigma}{\partial y^\nu} + \frac{\partial h^{kj}_\nu}{\partial y^\sigma}\right) y^\nu_k \tag{3.40}$$

is a local coordinate transformation on J^1Y.

Since

$$\theta_\lambda = -H\omega_0 + p^j_\sigma\, \mathrm{d}y^\sigma \wedge \omega_j + d(h^{ji}_\sigma y^\sigma_j \omega_i), \tag{3.41}$$

where

$$H = -L_0 + \frac{\partial L_0}{\partial y^\sigma_j} y^\sigma_j - \frac{\partial h^{jk}_\sigma}{\partial y^\nu} y^\sigma_j y^\nu_k, \tag{3.42}$$

and the momenta p^j_σ are given by (3.40), one gets Hamilton–De Donder equations expressed in the *Legendre coordinates* (3.40) in the "familiar" canonical form

$$\frac{\partial y^\nu}{\partial x^k} = \frac{\partial H}{\partial p^k_\nu}, \qquad \frac{\partial p^i_\nu}{\partial x^i} = -\frac{\partial H}{\partial y^\nu}. \tag{3.43}$$

To summarize, Lagrangian systems defined by second-order Lagrangians (3.37) are naturally of the *first-order*. If, moreover, regularity is understood in a more general (and geometrical) way as a condition for one-to-one correspondence between extremals and Hamilton extremals, one obtains a new regularity condition (3.39) and formulas (3.42) and (3.40) for the Hamiltonian and momenta, which differ from the usual ones, however, contain the standard formulas for first-order Lagrangians as a special case.

As pointed out by Krupka and Štěpánková, the above results directly apply to the *Einstein–Hilbert Lagrangian* (scalar curvature) of the general relativity theory (for explicit computations see [43], cf. also [28]). Thus, within this setting, gravity naturally appears as a *first-order regular* theory (without constraints).

Later the above ideas were applied to study also some other kinds of higher-order Lagrangians with projectable Poincaré–Cartan forms by Garcia and Muñoz Masqué [16] (cf. also comments in [17]).

4. A new look at Hamilton field theory

Now, we are in position to explain our setting for Hamiltonian field theories on fibred manifolds. Inspired by Dedecker's approach to Hamilton equations [5], and Krupka's theory of Lepagean n-forms [31,35,41], our approach is a straightforward generalization to field theory of ideas developed within *higher-order mechanics*, and *based upon the so-called Lepagean* 2*-forms* ($2 = \dim X + 1$) [44–46].

4.1. Lagrangian and Hamiltonian systems

A $(n+1)$-form E on J^sY, $s \geq 1$, is called a *dynamical form* if it is 1-contact and $\pi_{s,0}$-horizontal. This means that E is a dynamical form iff in every fibred chart

$$E = E_\sigma \omega^\sigma \wedge \omega_0, \tag{4.1}$$

where E_σ are functions on $V_s \subset J^sY$. A section γ of π is called a *path* of E if $E \circ J^s\gamma = 0$ (E considered as a section of the bundle $\Lambda^{n+1}(J^sY) \to J^sY$). In fibred coordinates this equation represents a *system of m partial differential equations of order s*,

$$E_\sigma(x^i, \gamma^\nu, D_j\gamma^\nu, \dots, D_{j_1\cdots j_s}\gamma^\nu) = 0 \tag{4.2}$$

for the components $\gamma^\nu(x^i)$, $1 \le \nu \le m$, of γ.

The key-concept in the present approach is that of a *Lepagean* $(n+1)$-*form*.

Definition 4.1. Let $s \ge 0$. A *closed* $(n+1)$-*form* α on J^sY will be called *Lepagean* if $p_1\alpha$ is a dynamical form.

In what follows, let us denote

$$p_1\alpha = E. \tag{4.3}$$

We can see that $E \in \Lambda_Y^{n+1}(J^{s+1}Y)$.

If α is a Lepagean $(n+1)$-form and $E = p_1\alpha$, we also say that α is a *Lepagean equivalent of* E.

With help of the definition of a Lepagean n-form and the Poincaré Lemma the following proposition is obtained immediately.

Proposition 4.1.

(1) *Every Lepagean* $(n+1)$-*form locally equals to the exterior derivative of a Lepagean n-form*.
(2) *The* 1-*contact part E of a Lepagean* $(n+1)$-*form is a locally variational form* (*i.e., there exists an open covering of* $J^{s+1}Y$ *such that, on each set of this covering, E coincides with the Euler–Lagrange form of a Lagrangian*).
(3) *If* α *is a Lepagean* $(n+1)$-*form then the equations for paths of* $E = p_1\alpha$ *are the Euler–Lagrange equations*.
(4) *If* α *is a Lepagean* $(n+1)$-*form then the components* E_σ *of* $E = p_1\alpha$ *satisfy the identities*

$$\frac{\partial E_\sigma}{\partial y^\nu_{j_1 j_2\cdots j_l}} - \sum_{k=l}^{s+1}(-1)^l \binom{k}{l} d_{j_{l+1}} d_{j_{l+2}} \cdots d_{j_k} \frac{\partial E_\nu}{\partial y^\sigma_{j_1 j_2\cdots j_l}} = 0, \quad 0 \le l \le s+1. \tag{4.4}$$

Recall that (4.4) are necessary and sufficient conditions for local variationality of a dynamical form; they are called *Anderson–Duchamp–Krupka conditions* [2,34].

We say that two Lepagean $(n+1)$-forms α_1 and α_2 (possibly of different orders) are *equivalent* if (up to a possible projection)

$$p_1\alpha_1 = p_1\alpha_2. \tag{4.5}$$

The equivalence class of α will be denoted by $[\alpha]$.

Definition 4.2. The class $[\alpha]$ of all equivalent Lepagean $(n+1)$-forms is called a *Lagrangian system*. Paths of a Lagrangian system are called *extremals*.

Note that the class $[\alpha]$ contains *all* Lepagean equivalents of the locally variational form $E = p_1\alpha$. This means that the class $[\alpha]$ is a representative of the family of *all equivalent Lagrangians* whose Euler–Lagrange form (possibly locally) coincides with E.

Proposition 4.2. *Let* $[\alpha]$ *be a Lagrangian system,* $E = p_1\alpha$ *the corresponding dynamical form. Let* $s \geq 0$ *denote the minimum of the set of orders of the forms belonging to the class* $[\alpha]$. *The following conditions are equivalent:*

(1) *A section* $\gamma : U \to Y$ *defined on an open subset U of X is an extremal of E.*
(2) *For every* π*-vertical vector field* ξ *on Y,*

$$J^s\gamma^* i_{J^s\xi}\alpha = 0, \tag{4.6}$$

where α *is any representative of order s of the equivalence class* $[\alpha]$.

Proof. Suppose (1). Then, by definition, $E \circ J^{s+1}\gamma = 0$, i.e., $E_\sigma \circ J^{s+1}\gamma = 0$, $1 \leq \sigma \leq m$. This means that for every π-vertical vector field ξ on Y,

$$J^{s+1}\gamma^* i_{J^{s+1}\xi}E = J^{s+1}\gamma^*((E_\sigma\xi^\sigma)\omega_0) = ((E_\sigma\xi^\sigma)\circ J^{s+1}\gamma)\omega_0 = 0.$$

Hence, we get for every π-vertical vector field ξ on Y, and every $\alpha \in [\alpha]$ such that α is defined on J^sY,

$$\begin{aligned} J^s\gamma^* i_{J^s\xi}\alpha &= J^{s+1}\gamma^* i_{J^{s+1}\xi}\pi^*_{s+1,s}\alpha = J^{s+1}\gamma^* i_{J^{s+1}\xi}\pi^*_{s+1,s}(p_1\alpha) \\ &= J^{s+1}\gamma^* i_{J^{s+1}\xi}E = 0. \end{aligned}$$

Conversely, suppose that γ satisfies Eq. (4.6). Taking (any) $\alpha \in [\alpha]$ defined on J^sY, and using $E = p_1\alpha$, we get by similar arguments as above, $E \circ J^{s+1}\gamma = 0$. □

Accordingly, (4.6) are called *Euler–Lagrange equations* corresponding to the Lagrangian system $[\alpha]$.

Remark 4.1 (On the order of a Lagrangian system). Let us stop for a moment to discuss the concept of the *order* of a Lagrangian system. First, note that usually a Lagrangian system of order r is identified with a *global Lagrangian on* J^rY. Here, Definition 4.2 of a Lagrangian system is more general. It means in fact that a Lagrangian system is *a family of all equivalent Lagrangians* which give rise to an Euler–Lagrange form. It should be stressed that this family contains *Lagrangians of all orders starting from a certain minimal one* which are *defined on open subsets* of the corresponding jet prolongations of the fibred manifold $\pi : Y \to X$. Often, there exists *no* global Lagrangian: obstructions lie in the topology of Y. Even if a global Lagrangian does exist, it is known that its order equals to the order of the corresponding Euler–Lagrange form. The question under what conditions a global Lagrangian is *globally reducible* to a minimal order Lagrangian (i.e., under what conditions there exists a global Lagrangian of the minimal order for E), is still open (for more details see [2,25,42]).

Since in a general situation a Lagrangian system is characterized rather by a family of local Lagrangians of different orders than by a distinguished global minimal order Lagrangian, the above understanding of a Lagrangian system as an *equivalence class of Lepagean* $(n+1)$*-forms*, becomes quite natural. However, one has to precise the concept of the *order* of a Lagrangian system. Apparently, with a Lagrangian system two characteristic numbers are associated:

(i) The minimum of the set of orders of the forms belonging to the class $[\alpha]$: by Proposition 4.2, this number, s, characterizes the jet prolongation, J^sY, where the *dynamics* proceeds, and is directly related with the "true" order of the Euler–Lagrange form. The corresponding Euler–Lagrange equations are PDE of order $s+1$. We shall call s the *dynamical order* of the Lagrangian system $[\alpha]$.
(ii) The minimum of the set of orders of all Lagrangians giving rise to the Lagrangian system $[\alpha]$: This number, r_0, will be called the *order* of the Lagrangian system $[\alpha]$. Note that if s is the dynamical order of $[\alpha]$, one has $s \leq 2r_0 - 1$.

Note that the above definitions are concerned merely with characteristics directly referring to dynamics, hence common to equivalent Lagrangians, while distinct properties of particular Lagrangians which are not essential for the dynamics are eliminated.

In view of the above remarks, in particular, by a *first-order Lagrangian system* we shall mean a *family of* (*local*) *equivalent first-order Lagrangians on* J^1Y, or, equivalently, an *Euler–Lagrange form possessing* (*local*) *first-order Lagrangians*. Note that this means that a first-order Lagrangian system is either of the *dynamical order* 0, corresponding to E on J^1Y, or 1, corresponding to E defined on J^2Y (and not projectable onto J^1Y).

Definition 4.3. By a *Hamiltonian system of order s*, we shall mean a Lepagean $(n+1)$-form α on J^sY. A section δ of the fibred manifold π_s is called a *Hamilton extremal of* α if

$$\delta^* i_\xi \alpha = 0 \quad \text{for every } \pi_s\text{-vertical vector field } \xi \text{ on } J^sY. \tag{4.7}$$

Eq. (4.7) will be then called *Hamilton equations* of α.

Note that Hamilton equations are not uniquely determined by an Euler–Lagrange form (respectively, by a Lagrangian) but depend upon the form $\pi^*_{s+1,s}\alpha - E$, i.e., the part of α which is *at least* 2*-contact*. Consequently, one has *many different "Hamilton theories" associated to a given variational problem*.

On the other hand, we can see that two different *Lepagean n-forms* ρ_1 and ρ_2 (possibly of different orders) give rise to the *same* Hamiltonian system whenever $d\rho_1 = d\rho_2$, i.e., locally, $\rho_2 = \rho_1 + d\eta$. In this sense, we can understand a Hamiltonian system to be the equivalence class of (generally locally defined) *Lepagean n-forms*, differing by closed $(n-1)$-forms, and we have the following terminology.

Definition 4.4. Let $1 \leq i \leq n$. If in a neighbourhood of every point in J^sY there exists an *at most i-contact* Lepagean n-form ρ such that $\alpha = d\rho$, we call the corresponding Hamilton Eq. (4.7) *Hamilton* p_i*-equations*, and we speak about *Hamilton* p_i*-theory*.

In particular, *Hamilton* p_1*-equations* are locally based upon the Poincaré–Cartan form Θ, i.e., they are the familiar Hamilton–De Donder equations.

Hamilton p_2-equations are locally based upon a Lepagean form $\rho = \Theta + \mu_2$, where μ_2 is 2-contact. Hamilton p_2-equations related with first-order Lagrangians have been studied in [49,50], second-order Lagrangians are discussed in [58].

Hamilton p_n-equations are locally based upon a general Lepagean n-form. A first-order case (on manifolds of contact elements) was studied by Dedecker [5] (recall Section 3.4). Higher-order Hamilton equations of this kind (on fibred manifolds) appear in [36], however, only the Hamilton–De Donder case $\rho = \Theta$ is discussed.

For a Hamiltonian system α of order $s \geq 1$, the sets of extremals and Hamilton extremals need not be in bijective correspondence, i.e., a Hamiltonian system may possess Hamilton extremals which are not prolongations of extremals. Moreover, in general, not every Hamilton extremal projects onto an extremal. However, it is easy to show the following relations between the sets of extremals and Hamilton extremals.

Proposition 4.3. *Let $[\alpha]$ be a Lagrangian system, $E = p_1\alpha$ the related locally variational form.*

(1) *If γ is an extremal of E then for every Lepagean equivalent α of E, the section $\delta = J^s\gamma$ (where s is the order of α) is a Hamilton extremal of α. Conversely, if α is a Lepagean equivalent of E defined on J^sY and δ is a holonomic Hamilton extremal of α then $\gamma = \pi_{s,0} \circ \delta$ is an extremal of E.*
(2) *For α defined on J^sY, the map J^s is a bijection between the set of extremals of E and the set of holonomic Hamilton extremals of α.*

Proof. Both the assertions in (1) follow directly from the fact that for $\delta = J^s\gamma$ the equation $J^s\gamma^* i_\xi \alpha = 0$ (for every π_s-vertical ξ on J^sY) depends only upon the projection $T\pi_{s,0} \cdot \xi$ of ξ onto Y.

Let us show (2). By the second part of (1), J^s is surjective. It is also injective, since if for two holonomic Hamilton extremals, $\delta_1 = J^s\gamma_1$ and $\delta_2 = J^s\gamma_2$ it holds $\delta_1 = \delta_2$, we get by (1), $\gamma_1 = \pi_{s,0} \circ \delta_1 = \pi_{s,0} \circ \delta_2 = \gamma_2$. □

Remark 4.2. Let us mention the geometric meaning of the Hamilton equation (4.7). Denote

$$\mathcal{D}^s_\alpha = \{i_\xi \alpha \,|\, \text{where } \xi \text{ runs over all } \pi_s\text{-vertical vector fields on } J^sY\}. \quad (4.8)$$

We call the ideal of differential forms on J^sY generated by the system of n-forms $\mathcal{D}^s_\alpha$ the *Hamiltonian ideal* related with α. Now, Eq. (4.7) means that Hamilton extremals identify with *integral sections* of the Hamiltonian ideal. From the point of view of the geometric theory of differential equations, this is an extremely important property, pointing out the *geometric content* of Hamilton theory *in contrast with a usual understanding it merely as a certain "formalism"* in the calculus of variations. Moreover, in view of the above proposition, if $[\alpha]$ is a Lagrangian system, each of its associated Hamiltonian systems (i.e., Lepagean $(n+1)$-forms belonging to the class $[\alpha]$) can be viewed as a different *extension* of the original variational problem. Consequently, in any concrete situation one can utilize the possibility to apply additional requirements (geometrical and/or physical) to choose from many alternative Hamiltonian systems related with a given Lagrangian system the "most appropriate" one. This will be our task in the next sections.

4.2. First-order Hamiltonian systems

In the sequel of this paper, we shall study in detail the case of Lepagean $(n+1)$-forms defined on J^1Y. These Hamiltonian systems are the most simple ones from the mathematical point of view, and, moreover, *from the physical point of view they represent Hamiltonian counterparts of all the most interesting Lagrangian systems in field theory*. For higher-order generalizations we refer to [47,48].

Let α be a Lepagean $(n+1)$-form on J^1Y. Using the canonical decomposition of α into the sum of i-contact components, $1 \leq i \leq n+1$, we write

$$\pi^*_{2,1}\alpha = E + F + G,$$

where $E = p_1\alpha$ (as above), $F = p_2\alpha$, and G is *at least* 3-contact. We also set

$$\hat{\alpha} = E + F, \tag{4.9}$$

and call $\hat{\alpha}$ the *principal part of* α. Note that $\hat{\alpha}$ is an $(n+1)$-form on J^2Y, generally *not closed*.

Theorem 4.1. *Let α be a Lepagean $(n+1)$-form on J^1Y. The following two assertions are equivalent*:

(1) $p_2\,\mathrm{d}\alpha = 0$.
(1) *E satisfies the Anderson–Duchamp–Krupka conditions, i.e.*,

$$\frac{\partial E_\sigma}{\partial y^\nu} - \frac{\partial E_\nu}{\partial y^\sigma} + d_j\frac{\partial E_\nu}{\partial y^\sigma_j} - d_jd_k\frac{\partial E_\nu}{\partial y^\sigma_{jk}} = 0, \qquad \frac{\partial E_\sigma}{\partial y^\nu_j} + \frac{\partial E_\nu}{\partial y^\sigma_j} - 2d_k\frac{\partial E_\nu}{\partial y^\sigma_{jk}} = 0,$$
$$\frac{\partial E_\sigma}{\partial y^\nu_{jk}} - \frac{\partial E_\nu}{\partial y^\sigma_{jk}} = 0, \tag{4.10}$$

and F takes the form

$$F = \left(\frac{1}{4}\left(\frac{\partial E_\sigma}{\partial y^\nu_i} - \frac{\partial E_\nu}{\partial y^\sigma_i}\right) - d_j f^{i,j}_{\sigma\nu}\right)\omega^\sigma\wedge\omega^\nu\wedge\omega_i + \left(\frac{\partial E_\sigma}{\partial y^\nu_{ij}} - 2f^{j,i}_{\nu\sigma}\right)\omega^\sigma$$
$$\wedge\,\omega^\nu_j\wedge\omega_i + f^{jk,i}_{\sigma\nu}\omega^\sigma_j\wedge\omega^\nu_k\wedge\omega_i, \tag{4.11}$$

where

$$f^{j,k}_{\sigma\nu} - f^{k,j}_{\nu\sigma} - d_i f^{ik,j}_{\sigma\nu} = 0, \tag{4.12}$$

and $f^{i,j}_{\sigma\nu}$, $f^{jk,i}_{\sigma\nu}$ are arbitrary functions satisfying the antisymmetry relations

$$f^{j,k}_{\sigma\nu} = -f^{k,j}_{\sigma\nu}, \qquad f^{ki,j}_{\sigma\nu} = -f^{ji,k}_{\sigma\nu}, \qquad f^{ki,j}_{\sigma\nu} = -f^{ik,j}_{\nu\sigma}. \tag{4.13}$$

Proof. Taking $\hat{\alpha}$ in the form (4.9), denote

$$E = E_\sigma\omega^\sigma\wedge\omega_0,$$
$$F = F^{\,i}_{\sigma\nu}\omega^\sigma\wedge\omega^\nu\wedge\omega_i + 2F^{j,i}_{\sigma\nu}\omega^\sigma_j\wedge\omega^\nu\wedge\omega_i + F^{jk,i}_{\sigma\nu}\omega^\sigma_j\wedge\omega^\nu_k\wedge\omega_i, \tag{4.14}$$

where

$$F_{\sigma\nu}^{,i} = -F_{\nu\sigma}^{,i}, \qquad F_{\sigma\nu}^{jk,i} = -F_{\nu\sigma}^{kj,i}. \tag{4.15}$$

We get

$$\begin{aligned} p_2\,\mathrm{d}E &= \frac{\partial E_\sigma}{\partial y^\nu}\omega^\nu \wedge \omega^\sigma \wedge \omega_0 + \frac{\partial E_\sigma}{\partial y_k^\nu}\omega_k^\nu \wedge \omega^\sigma \wedge \omega_0 + \frac{\partial E_\sigma}{\partial y_{kl}^\nu}\omega_{kl}^\nu \wedge \omega^\sigma \wedge \omega_0, \\ p_2\,\mathrm{d}F &= d_i F_{\sigma\nu}^{,i}\omega^\sigma \wedge \omega^\nu \wedge \omega_0 + 2(d_i F_{\sigma\nu}^{j,i} + F_{\sigma\nu}^{,j})\omega_j^\sigma \wedge \omega^\nu \wedge \omega_0 \\ &\quad + (d_i F_{\sigma\nu}^{jk,i} + 2F_{\sigma\nu}^{j,k})\omega_j^\sigma \wedge \omega_k^\nu \wedge \omega_0 + 2F_{\sigma\nu}^{j,i}\omega_{ji}^\sigma \wedge \omega^\nu \wedge \omega_0 \\ &\quad + 2F_{\sigma\nu}^{jk,i}\omega_{ji}^\sigma \wedge \omega_k^\nu \wedge \omega_0. \end{aligned}$$

Condition (1) of the theorem means $p_2\,\mathrm{d}E + p_2\,\mathrm{d}F = 0$, and we get the following identities:

$$\begin{aligned} &\frac{1}{2}\left(\frac{\partial E_\nu}{\partial y^\sigma} - \frac{\partial E_\sigma}{\partial y^\nu}\right) + d_i F_{\sigma\nu}^{,i} = 0, \quad \frac{\partial E_\nu}{\partial y_j^\sigma} + 2d_i F_{\sigma\nu}^{j,i} + 2F_{\sigma\nu}^{,j} = 0, \\ &d_i F_{\sigma\nu}^{jk,i} + F_{\sigma\nu}^{j,k} - F_{\nu\sigma}^{k,j} = 0, \quad \frac{\partial E_\nu}{\partial y_{ji}^\sigma} + F_{\sigma\nu}^{j,i} + F_{\sigma\nu}^{i,j} = 0, \quad F_{\sigma\nu}^{jk,i} + F_{\sigma\nu}^{ik,j} = 0. \end{aligned} \tag{4.16}$$

Hence,

$$(F_{\sigma\nu}^{ik,j})_{\mathrm{sym}(ij)} = 0, \qquad (F_{\sigma\nu}^{i,j})_{\mathrm{sym}(ij)} = -\frac{1}{2}\frac{\partial E_\nu}{\partial y_{ji}^\sigma}, \tag{4.17}$$

where sym(ij) means symmetrization in the indicated indices. Let us denote by $f_{\sigma\nu}^{ik,j}$, and $f_{\sigma\nu}^{i,j}$ the antisymmetric part with respect to the indices i, j of the $F_{\sigma\nu}^{ik,j}$ and $F_{\sigma\nu}^{i,j}$, respectively. Then the third equation of (4.16) takes the form

$$d_i f_{\sigma\nu}^{jk,i} + (F_{\sigma\nu}^{j,k})_{\mathrm{sym}(jk)} + f_{\sigma\nu}^{j,k} - (F_{\nu\sigma}^{k,j})_{\mathrm{sym}(jk)} - f_{\nu\sigma}^{k,j} = 0,$$

i.e., it splits (by taking its symmetric and antisymmetric part in j, k) to the following two relations:

$$\begin{aligned} &\tfrac{1}{2}d_i(f_{\sigma\nu}^{jk,i} + f_{\sigma\nu}^{kj,i}) + (F_{\sigma\nu}^{j,k})_{\mathrm{sym}(jk)} - (F_{\nu\sigma}^{k,j})_{\mathrm{sym}(jk)} = 0, \\ &\tfrac{1}{2}d_i(f_{\sigma\nu}^{jk,i} - f_{\sigma\nu}^{kj,i}) + f_{\sigma\nu}^{j,k} - f_{\nu\sigma}^{k,j} = 0. \end{aligned} \tag{4.18}$$

Since, however,

$$f_{\sigma\nu}^{jk,i} + f_{\sigma\nu}^{kj,i} = -f_{\sigma\nu}^{ik,j} - f_{\sigma\nu}^{ij,k} = f_{\nu\sigma}^{ki,j} + f_{\nu\sigma}^{jk,i} = 0,$$

the first of the above relations becomes

$$(F_{\sigma\nu}^{j,k})_{\mathrm{sym}(jk)} - (F_{\nu\sigma}^{k,j})_{\mathrm{sym}(jk)} = 0,$$

i.e., by (4.17),

$$\frac{\partial E_\sigma}{\partial y_{jk}^\nu} - \frac{\partial E_\nu}{\partial y_{jk}^\sigma} = 0,$$

which is one of the variationality conditions (4.10). In the second of the relations (4.18), we recognize (4.12).

Now, let us consider the second of the Eq. (4.16). It splits into two relations, the symmetrized and antisymmetrized one in σ, ν, respectively. Computing the symmetrized part, we get using (4.17) and (4.12),

$$\begin{aligned}-\frac{1}{2}\left(\frac{\partial E_\sigma}{\partial y^\nu_j}+\frac{\partial E_\nu}{\partial y^\sigma_j}\right) &= d_i((F^{j,i}_{\sigma\nu})_{\mathrm{sym}(ij)}+(F^{j,i}_{\nu\sigma})_{\mathrm{sym}(ij)}+f^{j,i}_{\sigma\nu}+f^{j,i}_{\nu\sigma})\\ &= -\frac{1}{2}d_i\left(\frac{\partial E_\nu}{\partial y^\sigma_{ij}}+\frac{\partial E_\sigma}{\partial y^\nu_{ij}}\right)+d_i(f^{j,i}_{\sigma\nu}+f^{i,j}_{\sigma\nu})-d_i d_k f^{kj,i}_{\sigma\nu}\\ &= -d_i\frac{\partial E_\nu}{\partial y^\sigma_{ij}},\end{aligned}$$

i.e., the second of the variationality conditions. The corresponding antisymmetrized part becomes

$$F^{\ j}_{\sigma\nu}=\frac{1}{4}\left(\frac{\partial E_\sigma}{\partial y^\nu_j}-\frac{\partial E_\nu}{\partial y^\sigma_j}\right)-\frac{1}{2}d_i(f^{j,i}_{\sigma\nu}-f^{i,j}_{\sigma\nu})-\frac{1}{2}d_i d_k f^{kj,i}_{\sigma\nu},$$

and since the last term, $d_i d_k f^{kj,i}_{\sigma\nu}$, equals 0,

$$F^{\ j}_{\sigma\nu}=\frac{1}{4}\left(\frac{\partial E_\sigma}{\partial y^\nu_j}-\frac{\partial E_\nu}{\partial y^\sigma_j}\right)-d_i f^{j,i}_{\sigma\nu}. \tag{4.19}$$

Collecting (4.17) and (4.19), we can see that F is of the form (4.11), as desired.

Finally, one has to utilize the first relation of (4.16). However, substituting (4.19) and using the second and third relation of (4.10), we easily obtain

$$\begin{aligned}0&=\frac{\partial E_\nu}{\partial y^\sigma}-\frac{\partial E_\sigma}{\partial y^\nu}+\frac{1}{2}d_j\left(\frac{\partial E_\sigma}{\partial y^\nu_j}-\frac{\partial E_\nu}{\partial y^\sigma_j}\right)+2d_j d_i f^{j,i}_{\sigma\nu}\\ &=-\left(\frac{\partial E_\sigma}{\partial y^\nu}-\frac{\partial E_\nu}{\partial y^\sigma}+d_j\frac{\partial E_\nu}{\partial y^\sigma_j}-d_j d_k\frac{\partial E_\nu}{\partial y^\sigma_{jk}}\right)\end{aligned}$$

which is the first of (4.10), and we are done.

Conversely, taking F in the form (4.11), computing $p_1\,\mathrm{d}E+p_2\,\mathrm{d}F$, and using (4.10), (4.12) and (4.13), we arrive at $p_2\,\mathrm{d}\alpha=0$. □

Remark 4.3. Note that the 2-contact part F of α (given by the formulas (4.11)–(4.13)) can be expressed as follows:

$$F=F_E-p_2\,\mathrm{d}\phi,\qquad F_E=\frac{1}{4}\left(\frac{\partial E_\sigma}{\partial y^\nu_i}-\frac{\partial E_\nu}{\partial y^\sigma_i}\right)\omega^\sigma\wedge\omega^\nu\wedge\omega_i+\frac{\partial E_\sigma}{\partial y^\nu_{ij}}\omega^\sigma\wedge\omega^\nu_j\wedge\omega_i,$$

$$\phi=\frac{1}{2}(f^{i,j}_{\sigma\nu}\omega^\sigma\wedge\omega^\nu\wedge\omega_{ij}+f^{ik,j}_{\sigma\nu}\omega^\sigma\wedge\omega^\nu_k\wedge\omega_{ij}) \tag{4.20}$$

and the relations (4.12) and (4.13) hold.

Let us recall transformation properties of the contact forms ω^σ, ω^σ_j. If $(x^i, y^\sigma, y^\sigma_j)$ and $(\bar{x}^i, \bar{y}^\sigma, \bar{y}^\sigma_j)$ are fibred coordinates defined on an open subset of J^1Y, it holds

$$\bar{\omega}^\sigma = \frac{\partial \bar{y}^\sigma}{\partial y^\nu}\omega^\nu, \qquad \bar{\omega}^\sigma_j = \frac{\partial \bar{y}^\sigma_j}{\partial y^\nu_k}\omega^\nu_k + \frac{\partial \bar{y}^\sigma_j}{\partial y^\nu}\omega^\nu. \tag{4.21}$$

Taking into account these formulas, one can see immediately that in the class $[\alpha]$ there are distinguished elements as follows.

Corollary 4.1. *The forms F (4.11) with $f^{jk,i}_{\sigma\nu} = 0$ are invariant under fibred transformations*.

Consequently, to a Lagrangian system $[\alpha]$ one can associate a family of Hamiltonian systems with the 2-contact parts $F = F_E - p_2\,\mathrm{d}\phi$, where ϕ is $\pi_{2,0}$-horizontal, i.e.,

$$\phi = \tfrac{1}{2} f^{i,j}_{\sigma\nu}\omega^\sigma \wedge \omega^\nu \wedge \omega_{ij}, \quad f^{i,j}_{\sigma\nu} = -f^{j,i}_{\sigma\nu} = -f^{i,j}_{\nu\sigma}. \tag{4.22}$$

Corollary 4.2. *Let λ be a first-order Lagrangian. Then putting in* (4.11)

$$f^{jk,i}_{\sigma\nu} = 0, \qquad f^{i,j}_{\sigma\nu} = \frac{1}{4}\left(\frac{\partial^2 L}{\partial y^\sigma_i\,\partial y^\nu_j} - \frac{\partial^2 L}{\partial y^\sigma_j\,\partial y^\nu_i}\right), \tag{4.23}$$

one obtains a family of associated Hamiltonian systems such that

$$\hat{\alpha} = E + F = \mathrm{d}\theta_\lambda. \tag{4.24}$$

Proof. For the choice (4.23), the (anti)symmetry relations of (4.22) are obviously satisfied. Substituting (4.23) into (4.20) and using

$$E_\sigma = \frac{\partial L}{\partial y^\sigma} - d_j \frac{\partial L}{\partial y^\sigma_j},$$

we get

$$F = \frac{1}{2}\left(\frac{\partial^2 L}{\partial y^\sigma\,\partial y^\nu_i} - \frac{\partial^2 L}{\partial y^\nu\,\partial y^\sigma_i}\right)\omega^\sigma \wedge \omega^\nu \wedge \omega_i - \frac{\partial^2 L}{\partial y^\sigma_i\,\partial y^\nu_j}\omega^\sigma \wedge \omega^\nu_j \wedge \omega_i = p_2\,\mathrm{d}\theta_\lambda.$$

So, we are done. □

Remark 4.4. First-order Hamiltonian systems are associated with Euler–Lagrange forms defined on J^2Y. This means that the family of Lagrangian systems which admit a *first-order Hamiltonian* counterpart consists of two essentially different subfamilies:

(i) *First-order Lagrangian systems*—if the Euler–Lagrange form possesses (at least local) first-order Lagrangians. An important example of such a Lagrangian system (behind the scalar, spinor, electromagnetic, Yang–Mills and other physical fields) is *gravity*, usually represented by *scalar curvature* (which is a second-order Lagrangian, however, locally reducible to the first-order).

(ii) A class of *second-order Lagrangian systems*: in this case the minimal-order Lagrangians for E are nontrivially of order *two* (i.e., are not reducible to the first-order). This concerns for example all the second-order Euler–Lagrange expressions which are nonaffine in the second derivatives of the field variables.

Hence, Theorem 4.1 (respectively, formula (4.20)) covers all *second-order Lagrangian systems* with second-order Euler–Lagrange equations. However, in the case (i), we obtain some simplifications. To see this, notice that if λ is a first-order Lagrangian then the 2-contact part of every its *first-order Lepagean equivalent* ρ becomes

$$p_2\rho = g^{ij}_{\sigma\nu}\omega^\sigma \wedge \omega^\nu \wedge \omega_{ij} + h^{q,ij}_{\sigma\nu}\omega^\sigma \wedge \omega^\nu_q \wedge \omega_{ij} + h^{pq,ij}_{\sigma\nu}\omega^\sigma_p \wedge \omega^\nu_q \wedge \omega_{ij},$$

where

$$g^{ij}_{\sigma\nu} = -g^{ji}_{\sigma\nu} = -g^{ij}_{\nu\sigma}, \quad h^{q,ij}_{\sigma\nu} = -h^{q,ji}_{\sigma\nu}, \quad h^{q,ij}_{\sigma\nu} + h^{j,iq}_{\sigma\nu} = 0,$$
$$h^{pq,ij}_{\sigma\nu} = -h^{pq,ji}_{\sigma\nu} = -h^{qp,ij}_{\nu\sigma}, \quad h^{pq,ij}_{\sigma\nu} + h^{pj,iq}_{\sigma\nu} = 0.$$

In particular, one has Lepagean equivalents which are $\pi_{1,0}$-*horizontal*, i.e., $h^{q,ij}_{\sigma\nu} = h^{pq,ij}_{\sigma\nu} = 0$ (cf. (3.33)). Now, computing the principal part of the corresponding Hamiltonian system $\alpha = \mathrm{d}\rho$, we get

$$\begin{aligned}\hat{\alpha} = {} & \mathrm{d}\theta_\lambda + 2d_k g^{ik}_{\sigma\nu}\omega^\sigma \wedge \omega^\nu \wedge \omega_i + 2(2g^{ij}_{\sigma\nu} + d_k h^{j,ik}_{\sigma\nu})\omega^\sigma \wedge \omega^\nu_j \wedge \omega_i \\ & + 2(h^{k,ij}_{\sigma\nu} + d_p h^{jk,ip}_{\sigma\nu})\omega^\sigma_j \wedge \omega^\nu_k \wedge \omega_i.\end{aligned} \tag{4.25}$$

Comparing this formula with Theorem 4.1 gives us the following result.

Proposition 4.4. *Every first-order Hamiltonian system associated with a first-order Lagrangian system is of the form described by* Theorem 4.1, *where*

$$f^{jk,i}_{\sigma\nu} = h^{k,ij}_{\sigma\nu} + h^{j,ik}_{\nu\sigma} + 2d_p h^{jk,ip}_{\sigma\nu}, \qquad f^{j,i}_{\nu\sigma} = \frac{1}{2}\frac{\partial E_\sigma}{\partial y^\nu_{ij}} + \frac{1}{2}\frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu_j} - 2g^{ij}_{\sigma\nu} - d_k h^{j,ik}_{\sigma\nu}.$$

If, in particular, $\alpha = \mathrm{d}\rho$, *where* ρ *is* $\pi_{1,0}$-*horizontal, one has*

$$f^{jk,i}_{\sigma\nu} = 0, \quad f^{i,j}_{\sigma\nu} = \frac{3}{4}\frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu_j} + \frac{1}{4}\frac{\partial^2 L}{\partial y^\sigma_j \partial y^\nu_i} - 2g^{ij}_{\sigma\nu}.$$

4.3. *Regularity as a geometrical concept*

Definition 4.5. A section δ of the fibred manifold $\pi_s : J^sY \to X$ is called a *Dedecker's section* if $\delta^*\mu = 0$ for every at least 2-contact form μ on J^sY.

A Dedecker's section which is a Hamilton extremal of $\pi^*_{2,1}\alpha$ is called *Dedecker–Hamilton extremal* of α. Hamilton equations, considered as equations for Dedecker's sections, are called *Dedecker–Hamilton equations*.

Let us study relations between Hamilton extremals and Dedecker–Hamilton extremals of a Hamiltonian system.

Proposition 4.5. *Let α be a first-order Hamiltonian system. If $\hat{\delta}$ is a Dedecker–Hamilton extremal of α then $\delta = \pi_{2,1} \circ \hat{\delta}$ is a Hamilton extremal of α.*

Proof. For a π_1-vertical vector field ξ on J^1Y, denote by $\hat{\xi}$ a π_2-vertical, $\pi_{2,1}$-projectable vector field on J^2Y which projects onto ξ. Then $\delta^* i_\xi \alpha = (\pi_{2,1} \circ \hat{\delta})^* i_\xi \alpha = \hat{\delta}^* \pi_{2,1}^* i_\xi \alpha = \hat{\delta}^* i_{\hat{\xi}} \pi_{2,1}^* \alpha = 0$, proving our assertion. □

If $\hat{\alpha}$ is the principal part of a first-order Hamiltonian system α, denote by $\mathcal{D}_{\hat{\alpha}}$ the family of n-forms $i_\xi \hat{\alpha}$, where ξ runs over all π_2-vertical vector fields on J^2Y. It is clear that Dedecker–Hamilton extremals of α are those Dedecker's sections which are *integral sections* of the ideal generated by $\mathcal{D}_{\hat{\alpha}}$. Using (4.11), we immediately get that $\mathcal{D}_{\hat{\alpha}}$ is locally spanned by the following n-forms:

$$\eta_\rho^{pq} = i_{\partial/\partial y_{pq}^\rho} \hat{\alpha} = 0, \quad \eta_\rho^p = i_{\partial/\partial y_p^\rho} \hat{\alpha} = -\left(\frac{\partial E_\nu}{\partial y_{ip}^\rho} - 2 f_{\rho\nu}^{p,i}\right) \omega^\nu \wedge \omega_i - 2 f_{\nu\rho}^{jp,i} \omega_j^\nu \wedge \omega_i,$$

$$\eta_\rho = i_{\partial/\partial y^\rho} \hat{\alpha} = E_\rho \omega_0 + \frac{1}{2}\left(\frac{\partial E_\rho}{\partial y_i^\nu} - \frac{\partial E_\nu}{\partial y_i^\rho} - 4 d_j f_{\rho\nu}^{i,j}\right) \omega^\nu \wedge \omega_i + \left(\frac{\partial E_\rho}{\partial y_{ij}^\nu} - 2 f_{\nu\rho}^{j,i}\right) \omega_j^\nu \wedge \omega_i. \tag{4.26}$$

The (invariant) choice $f_{\sigma\nu}^{jk,i} = 0$ for $\hat{\alpha}$ then simplifies the η_ρ^p's to

$$\eta_\rho^p = -\left(\frac{\partial E_\nu}{\partial y_{pi}^\rho} - 2 f_{\nu\rho}^{i,p}\right) \omega^\nu \wedge \omega_i. \tag{4.27}$$

Definition 4.6. We call a Hamiltonian system α on J^1Y *regular* if $\operatorname{rank} \mathcal{D}_{\hat{\alpha}} = \operatorname{rank} V\pi_1$, and the system of local generators of $\mathcal{D}_{\hat{\alpha}}$ contains all the n-forms

$$\omega^\sigma \wedge \omega_i, \quad 1 \le \sigma \le m, \quad 1 \le i \le n. \tag{4.28}$$

We refer to (4.28) as local *canonical* 1-*contact* n-*forms* on J^1Y.

Note that by definition, every Dedecker–Hamilton extremal of a *regular* Hamiltonian system is *holonomic up to the first-order*, i.e.,

$$\pi_{2,1} \circ \hat{\delta} = J^1(\pi_{2,0} \circ \hat{\delta}). \tag{4.29}$$

Consequently, applying Proposition 4.3, we immediately get the following fundamental property of regular Hamiltonian systems.

Theorem 4.2. *Let α be a first-order Hamiltonian system. If α is regular then it holds*:

(1) *Every Dedecker–Hamilton extremal of α projects onto an extremal of $E = p_1\alpha$.*
(2) *The map J^1 is a bijection of the set of extremals of $E = p_1\alpha$ onto the set of $\pi_{2,1}$-projections of Dedecker–Hamilton extremals of E.*

Now, we shall be interested in finding explicit *regularity conditions* for first-order Hamiltonian systems.

Proposition 4.6. *Let α be a first-order Hamiltonian system. Assume that* $\operatorname{rank}\mathcal{D}_{\hat\alpha} = \operatorname{rank} V\pi_1$, *and for every $\pi_{2,0}$-vertical vector field ξ on J^2Y the n-form $i_\xi\hat\alpha$ is $\pi_{2,0}$-horizontal. Then, α is regular*.

Proof. By (4.26), the condition that for every $\pi_{2,0}$-vertical vector field ξ on J^2Y, $i_\xi\hat\alpha$ is $\pi_{2,0}$-horizontal means that all the functions $f^{jk,i}_{\sigma\nu}$ are equal 0, i.e., $\mathcal{D}_{\hat\alpha}$ contains the forms (4.27). The condition $\operatorname{rank}\mathcal{D}_{\hat\alpha} = m(n+1)$ then implies that the forms (4.27) are linearly independent. Hence, all the forms $\omega^\sigma \wedge \omega_i$ belong to $\mathcal{D}_{\hat\alpha}$, proving that α is regular. □

In keeping notations of Theorem 4.1, we have the following result.

Theorem 4.3. *Let α be a first-order Hamiltonian system. Suppose that*

$$f^{jk,i}_{\sigma\nu} = 0. \tag{4.30}$$

The following conditions are equivalent:

(1) *It holds*

$$\det\left(\frac{\partial E_\sigma}{\partial y^\nu_{ij}} - 2f^{i,j}_{\sigma\nu}\right) \neq 0, \tag{4.31}$$

where in the indicated $(mn\times mn)$-matrix, (σ, i) labels rows and (ν, j) labels columns.

(2) *α is regular*.

Proof. Taking into account (4.26), we can see that the assumptions (4.30) and (4.31) ensure that the forms η^p_ρ are independent, i.e., that $\mathcal{D}_{\hat\alpha}$ is locally generated by the forms

$$\omega^\sigma\wedge\omega_i,\quad E_\sigma\omega_0 + \left(\frac{\partial E_\sigma}{\partial y^\nu_{ij}} - 2f^{j,i}_{\nu\sigma}\right)\omega^\nu_j\wedge\omega_i,\quad 1\le\sigma\le m,\quad 1\le i\le n. \tag{4.32}$$

However, by (4.31), the matrix $(\partial E_\sigma/\partial y^\nu_{ij} - 2f^{i,j}_{\sigma\nu})$ with m rows labelled by σ and mn^2 columns labelled by (ν, j, i) has the maximal rank, m. Hence, $\operatorname{rank}\mathcal{D}_{\hat\alpha} = mn + m = \operatorname{rank} V\pi_1$, proving that α is regular.

Conversely, if $f^{jk,i}_{\sigma\nu} = 0$, we have $\mathcal{D}_{\hat\alpha}$ spanned by the system of n-forms

$$\left(\frac{\partial E_\nu}{\partial y^\rho_{ip}} - 2f^{p,i}_{\rho\nu}\right)\omega^\nu\wedge\omega_i,$$

$$E_\rho\omega_0 + \frac{1}{2}\left(\frac{\partial E_\rho}{\partial y^\nu_i} - \frac{\partial E_\nu}{\partial y^\rho_i} - 4d_j f^{i,j}_{\rho\nu}\right)\omega^\nu\wedge\omega_i + \left(\frac{\partial E_\rho}{\partial y^\nu_{ij}} - 2f^{j,i}_{\nu\rho}\right)\omega^\nu_j\wedge\omega_i.$$

By the regularity assumption, $\operatorname{rank}\mathcal{D}_{\hat\alpha} = \operatorname{rank} V\pi_1 = mn + m$, which means that all the above generators are independent. Consequently, (4.31) holds. □

Notice that the domain of definition of the functions $f^{i,j}_{\sigma\nu}$ in (4.31) is an open subset of J^2Y.

The regularity condition (4.31) can be expressed in an equivalent form by means of Lagrangians. Taking into account that

$$E_\sigma = \frac{\partial L}{\partial y^\sigma} - d_j \frac{\partial L}{\partial y^\sigma_j} + d_j d_k \frac{\partial L}{\partial y^\sigma_{jk}},$$

we immediately get the following result.

Corollary 4.3. *In terms of (any local) second-order Lagrangian for $E = p_1\alpha$, the condition (4.31) is equivalent with*

$$\det\left(\frac{\partial^2 L}{\partial y^\sigma \partial y^\nu_{pq}} + \frac{\partial^2 L}{\partial y^\nu \partial y^\sigma_{pq}} - \frac{\partial^2 L}{\partial y^\sigma_{(p} \partial y^\nu_{q)}} + d_j\left(2\frac{\partial^2 L}{\partial y^\nu_{(p} \partial y^\sigma_{q)j}} - \frac{\partial^2 L}{\partial y^\sigma_j \partial y^\nu_{pq}}\right) + d_j d_k \frac{\partial^2 L}{\partial y^\sigma_{jk} \partial y^\nu_{pq}} - 2 f^{p,q}_{\sigma\nu}\right) \neq 0,$$

where (p, q) means symmetrization in the indicated indices.

If, in particular, E defines a first-order Lagrangian system, then (4.31) can be expressed by means of (any local) first-order Lagrangian for E and takes the form

$$\det\left(\frac{\partial^2 L}{\partial y^\sigma_p \partial y^\nu_q} + \frac{\partial^2 L}{\partial y^\sigma_q \partial y^\nu_p} + 4 f^{p,q}_{\sigma\nu}\right) \neq 0.$$

Hence, using the notations of (4.25), and taking into account Proposition 4.4, we can see that by Theorem 4.3, the following corollary can be stated.

Corollary 4.4. *A first-order Hamiltonian system corresponding to a first-order Lagrangian system is regular if and only if the following conditions are satisfied*:

$$h^{k,ij}_{\sigma\nu} + h^{j,ik}_{\nu\sigma} + 2d_p h^{jk,ip}_{\sigma\nu} = 0, \qquad \det\left(\frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu_j} - 4g^{ij}_{\sigma\nu} - 2d_k h^{j,ik}_{\sigma\nu}\right) \neq 0.$$

In this way, we obtain the following assertions in the first of which we recognize *Dedecker's regularity condition* (cf. Theorem 3.5), and the second one is a generalization of *Krupka–Štěpánková regularity condition* (Theorem 3.6).

Corollary 4.5. *Let α be a first-order Hamiltonian system such that $E = p_1\alpha$ defines a first-order Lagrangian system. Suppose that (at least locally) $\alpha = \mathrm{d}\rho$, where ρ is $\pi_{1,0}$-horizontal.*

(1) *Let $\lambda = L\omega_0$ be a first-order Lagrangian for E. In terms of L, the regularity condition (4.31) reads*

$$\det\left(\frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu_j} - \Lambda^{ij}_{\sigma\nu}\right) \neq 0, \tag{4.33}$$

where $\Lambda^{ij}_{\sigma\nu}(x^k, y^\rho, y^\rho_p)$ are functions satisfying $\Lambda^{ij}_{\sigma\nu} = -\Lambda^{ji}_{\sigma\nu} = -\Lambda^{ij}_{\nu\sigma}$.

(2) *Let* $\lambda = L\omega_0$ *be a second-order Lagrangian for E. Denote* $L = L_0 + h^{kl}_{\rho} y^{\rho}_{kl}$, *where the functions* L_0 *and* h^{kl}_{ρ} *do not depend on the* y^{σ}_{pq}'s. *In terms of L the regularity condition* (4.31) *takes the form*

$$\det\left(\frac{\partial^2 L_0}{\partial y^{\sigma}_i \partial y^{\nu}_j} - \tilde{d}_k \frac{\partial h^{ik}_{\sigma}}{\partial y^{\nu}_j} - \frac{\partial h^{ij}_{\sigma}}{\partial y^{\nu}} - \frac{\partial h^{ji}_{\nu}}{\partial y^{\sigma}} - \Lambda^{ij}_{\sigma\nu}\right) \neq 0, \tag{4.34}$$

where $\Lambda^{ij}_{\sigma\nu}(x^k, y^{\rho}, y^{\rho}_p)$ *are functions satisfying* $\Lambda^{ij}_{\sigma\nu} = -\Lambda^{ji}_{\sigma\nu} = -\Lambda^{ij}_{\nu\sigma}$, *and* $\tilde{d}_k$ *denotes the operator* $\partial/\partial x^k + y^{\nu}_k \partial/\partial y^{\nu}$.

Proof.

(i) By Proposition 4.4 and formula (4.25), we have $f^{jk}_{\sigma\nu} = 0$, and

$$\hat{\alpha} = \mathrm{d}\theta_{\lambda} + 2d_k g^{ik}_{\sigma\nu}\omega^{\sigma} \wedge \omega^{\nu} \wedge \omega_i + 4g^{ij}_{\sigma\nu}\omega^{\sigma} \wedge \omega^{\nu}_j \wedge \omega_i.$$

Hence, (4.31) becomes the condition (4.33) with

$$\Lambda^{ij}_{\sigma\nu} = 4g^{ij}_{\sigma\nu} = \frac{1}{2}\left(\frac{\partial^2 L}{\partial y^{\sigma}_i \partial y^{\nu}_j} - \frac{\partial^2 L}{\partial y^{\sigma}_j \partial y^{\nu}_i}\right) - 2f^{i,j}_{\sigma\nu}. \tag{4.35}$$

(ii) If L_1 is a first-order Lagrangian equivalent with L, we have

$$L_1 = L - d_i f^i = L_0 - \tilde{d}_i f^i, \qquad h^{ki}_{\rho} = \frac{\partial f^i}{\partial y^{\rho}_k}.$$

Substituting L_1 into (4.33), we get (4.34). □

Recall that every Lagrangian λ on J^1Y has a (global) Lepagean equivalent

$$\rho^{\mathcal{K}}_{\lambda} = L\,\omega_0 + \sum_{k=1}^{n}\left(\frac{1}{k!}\right)^2 \frac{\partial^k L}{\partial y^{\sigma_1}_{j_1} \cdots \partial y^{\sigma_k}_{j_k}} \omega^{\sigma_1} \wedge \cdots \wedge \omega^{\sigma_k} \wedge \omega_{j_1\cdots j_k}. \tag{4.36}$$

This n-form, discovered by Krupka in 1977 ([33], cf. also [3]), is referred to as *Krupka form* of λ. It has the following important property: $\mathrm{d}\rho^{\mathcal{K}}_{\lambda} = 0$ *if and only if the Euler–Lagrange form* E_{λ} *identically vanishes*. Note that the Poincaré–Cartan form θ_{λ} does *not* possess a similar property.

Corollary 4.6. *Consider a Lagrangian* λ *on* J^1Y *and its Lepagean equivalent* $\rho^{\mathcal{K}}_{\lambda}$. *The Hamiltonian system* $\alpha = \mathrm{d}\rho^{\mathcal{K}}_{\lambda}$ *is regular if and only if*

$$\det\left(\frac{\partial^2 L}{\partial y^{\sigma}_i \partial y^{\nu}_j} + \frac{\partial^2 L}{\partial y^{\nu}_i \partial y^{\sigma}_j}\right) \neq 0, \quad \text{i.e.,}\ \det\left(\frac{\partial E_{\sigma}}{\partial y^{\nu}_{ji}}\right) \neq 0. \tag{4.37}$$

Proof. We can see immediately that for $\alpha = \mathrm{d}\rho_\lambda^{\mathcal{K}}$, (4.30) is satisfied. Hence, the assertion follows from Theorem 4.3 and Corollary 4.3, since in this case,

$$g_{\sigma\nu}^{ij} = \frac{1}{8}\left(\frac{\partial^2 L}{\partial y_i^\sigma \partial y_j^\nu} - \frac{\partial^2 L}{\partial y_i^\nu \partial y_j^\sigma}\right), \qquad \Lambda_{\sigma\nu}^{ij} = 4g_{\sigma\nu}^{ij} = \frac{1}{2}\left(\frac{\partial^2 L}{\partial y_i^\sigma \partial y_j^\nu} - \frac{\partial^2 L}{\partial y_i^\nu \partial y_j^\sigma}\right), \tag{4.38}$$

and by (4.34), $f_{\sigma\nu}^{i,j} = 0$. □

Note that since $f_{\sigma\nu}^{i,j} = 0$, the principal part $\hat{\alpha}$ of the Lepagean $(n+1)$-form $\alpha = \mathrm{d}\rho_\lambda^{\mathcal{K}}$ is (cf. (4.20))

$$\begin{aligned}\hat{\alpha} &= E_\sigma \omega^\sigma \wedge \omega_0 + \frac{1}{4}\left(\frac{\partial E_\sigma}{\partial y_i^\nu} - \frac{\partial E_\nu}{\partial y_i^\sigma}\right)\omega^\sigma \wedge \omega^\nu \wedge \omega_i + \frac{\partial E_\sigma}{\partial y_{ij}^\nu}\omega^\sigma \wedge \omega_j^\nu \wedge \omega_i \\ &= E + F_E.\end{aligned} \tag{4.39}$$

Another distinguished Lepagean n-form considered by Carathéodory [4] is

$$\begin{aligned}\rho_\lambda^{\mathcal{C}} &= \frac{1}{L^{n-1}}\left(L\,\mathrm{d}x^1 + \frac{\partial L}{\partial y_1^{\sigma_1}}\omega^{\sigma_1}\right) \wedge \cdots \wedge \left(L\,\mathrm{d}x^n + \frac{\partial L}{\partial y_n^{\sigma_n}}\omega^{\sigma_n}\right) \\ &= L\,\omega_0 + \frac{\partial L}{\partial y_j^\sigma}\omega^\sigma \wedge \omega_j + \frac{1}{2L}\frac{\partial L}{\partial y_j^\sigma}\frac{\partial L}{\partial y_k^\nu}\omega^\sigma \wedge \omega^\nu \wedge \omega_{jk} + \cdots .\end{aligned} \tag{4.40}$$

In this case, we immediately obtain the following corollary.

Corollary 4.7. *Consider a Lagrangian λ on J^1Y and its Lepagean equivalent $\rho_\lambda^{\mathcal{C}}$. The Hamiltonian system $\alpha = \mathrm{d}\rho_\lambda^{\mathcal{C}}$ is regular if and only if*

$$\det\left(\frac{\partial^2 L}{\partial y_i^\sigma \partial y_j^\nu} - \frac{1}{4L}\left(\frac{\partial L}{\partial y_i^\sigma}\frac{\partial L}{\partial y_j^\nu} - \frac{\partial L}{\partial y_j^\sigma}\frac{\partial L}{\partial y_i^\nu}\right)\right) \neq 0. \tag{4.41}$$

Remark 4.5. For a given Lagrangian system, formula (4.31) represents many "regularity conditions", dependent upon auxiliary parameters. As seen in Corollaries 4.5 and 4.6, one may choose in place of these parameters functions defined by a Lagrangian. In fact, at least locally, many different choices are possible, leading to various regularity conditions which involve only a Lagrangian. In this way, e.g., the regularity condition (4.33) covers the *"standard" regularity condition* (3.6) for $\Lambda_{\sigma\nu}^{ij} = 0$, the condition (4.37) for $\Lambda_{\sigma\nu}^{ij}$ given by (4.38), the Krupka–Štěpánková condition (3.39) for $\Lambda_{\sigma\nu}^{ij} = 0$, but also the condition (4.41), or, e.g., the following conditions [49]:

$$\det\left(k\frac{\partial^2 L}{\partial y_j^\sigma \partial y_i^\nu} - (k-1)\frac{\partial^2 L}{\partial y_i^\sigma \partial y_j^\nu}\right) \neq 0 \quad \text{for} \quad \Lambda_{\sigma\nu}^{ij} = k\left(\frac{\partial^2 L}{\partial y_i^\sigma \partial y_j^\nu} - \frac{\partial^2 L}{\partial y_j^\sigma \partial y_i^\nu}\right), \tag{4.42}$$

$$\det\left(\frac{\partial^2 L}{\partial y_i^\sigma \partial y_j^\nu} - \frac{\partial^3 L}{\partial x^i \partial y^\sigma \partial y_j^\nu} + \frac{\partial^3 L}{\partial x^j \partial y^\sigma \partial y_i^\nu} + \frac{\partial^3 L}{\partial x^i \partial y^\nu \partial y_j^\sigma} - \frac{\partial^3 L}{\partial x^j \partial y^\nu \partial y_i^\sigma}\right) \neq 0$$

$$\text{for}\quad \Lambda^{ij}_{\sigma\nu} = \frac{\partial^3 L}{\partial x^i \partial y^\sigma \partial y_j^\nu} - \frac{\partial^3 L}{\partial x^j \partial y^\sigma \partial y_i^\nu} - \frac{\partial^3 L}{\partial x^i \partial y^\nu \partial y_j^\sigma} + \frac{\partial^3 L}{\partial x^j \partial y^\nu \partial y_i^\sigma} \tag{4.43}$$

or

$$\det\left(\frac{\partial^2 L}{\partial y_i^\sigma \partial y_j^\nu} - \tilde{d}_i \frac{\partial^2 L}{\partial y^\sigma \partial y_j^\nu} + \tilde{d}_j \frac{\partial^2 L}{\partial y^\sigma \partial y_i^\nu} + \tilde{d}_i \frac{\partial^2 L}{\partial y^\nu \partial y_j^\sigma} - \tilde{d}_j \frac{\partial^2 L}{\partial y^\nu \partial y_i^\sigma}\right) \neq 0$$

$$\text{for}\quad \Lambda^{ij}_{\sigma\nu} = \tilde{d}_i \frac{\partial^2 L}{\partial y^\sigma \partial y_j^\nu} - \tilde{d}_j \frac{\partial^2 L}{\partial y^\sigma \partial y_i^\nu} - \tilde{d}_i \frac{\partial^2 L}{\partial y^\nu \partial y_j^\sigma} + \tilde{d}_j \frac{\partial^2 L}{\partial y^\nu \partial y_i^\sigma}. \tag{4.44}$$

Obviously, other conditions can be generated in a similar way.

5. Legendre transformation revisited

Consider a *regular* first-order Hamiltonian system α. Then, by definition, all the canonical 1-contact n-forms $\omega^\sigma \wedge \omega_i$ belong to the exterior differential system generated by $\mathcal{D}_{\hat{\alpha}}$. However, the generators of $\mathcal{D}_{\hat{\alpha}}$ naturally associated with fibred coordinates (i.e., (4.26)), are of the form of *linear combinations* of the $\omega^\sigma \wedge \omega_i$'s. In this sense fibred coordinates are not "canonical". In what follows our aim is to construct *new coordinates* in which *the forms* $\omega^\sigma \wedge \omega_i$ *appear as a part of the naturally associated generators*. Hence, (in the sense of the theory of differential systems) such coordinates are *adapted* to $\mathcal{D}_{\hat{\alpha}}$.

Definition 4.7. Let α be a *regular* Hamiltonian system on J^1Y such that $\hat{\alpha}$ is $\pi_{2,1}$-projectable. Let (W, χ), $\chi = (x^i, y^\sigma, p^i_\sigma)$ be a chart on J^1Y such that (x^i, y^σ) are local fibred coordinates on Y, and the canonical 1-contact n-forms $\omega^\sigma \wedge \omega_i$ coincide with the generators of $\mathcal{D}_{\hat{\alpha}}$ naturally associated with the coordinates p^i_σ, i.e.,

$$i_{\partial/\partial p^i_\sigma} \hat{\alpha} = \omega^\sigma \wedge \omega_i. \tag{4.45}$$

The chart (W, χ) will be called a *Legendre chart*, and $(x^i, y^\sigma, p^i_\sigma)$ are called *Legendre coordinates* associated with the regular Hamiltonian system α.

We shall study existence of Legendre charts. To this end, we keep notations used so far.

Theorem 4.4. *Let α be a regular Hamiltonian system on J^1Y associated with a first-order Lagrangian system. Let $x \in J^1Y$ be a point. Suppose that in a neighbourhood W of x,*

$$\alpha = \mathrm{d}\rho, \quad \rho = \theta_\lambda + \mu, \tag{4.46}$$

where λ is a Lagrangian for E defined on W, and $\pi^*_{2,1}\mu$ is an at least 2-contact n-form such that

$$p_2\mu = g^{ij}_{\sigma\nu}\omega^\sigma \wedge \omega^\nu \wedge \omega_{ij}, \quad g^{ij}_{\sigma\nu} = -g^{ji}_{\sigma\nu} = -g^{ij}_{\nu\sigma}. \tag{4.47}$$

where $g^{ij}_{\sigma\nu}$ are functions on $\pi_{1,0}(W)$. Put

$$p^i_\sigma = \frac{\partial L}{\partial y^\sigma_i} - 4g^{ij}_{\sigma\nu} y^\nu_j. \tag{4.48}$$

Then (W, χ), where $\chi = (x^i, y^\sigma, p^i_\sigma)$, is a Legendre chart for α.

Proof. First, notice that by assumptions, $\hat{\alpha}$ is projectable onto J^1Y. Next, we show that the matrix $(\partial p^i_\sigma / \partial y^\nu_k)$ is regular on W. From (4.48), we obtain

$$\frac{\partial p^i_\sigma}{\partial y^\nu_j} = \frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu_j} - 4g^{ij}_{\sigma\nu}. \tag{4.49}$$

Since α is regular, Corollary 4.5 gives that the above matrix is regular, proving that (W, χ), $\chi = (x^i, y^\sigma, p^i_\sigma)$ a chart on J^1Y.

It remains to show that condition (4.45) is satisfied. Using (4.48), we can write

$$\rho = -H\omega_0 + p^i_\sigma \,\mathrm{d}y^\sigma \wedge \omega_i + \eta + \mu_3, \tag{4.50}$$

where

$$\eta = g^{ij}_{\sigma\nu} \,\mathrm{d}y^\sigma \wedge \mathrm{d}y^\nu \wedge \omega_{ij}, \tag{4.51}$$

μ_3 is at least 3-contact, and

$$H = -L + p^i_\sigma y^\sigma_i + 2g^{ij}_{\sigma\nu} y^\sigma_i y^\nu_j. \tag{4.52}$$

Now,

$$\hat{\alpha} = -\mathrm{d}H \wedge \omega_0 + \mathrm{d}p^i_\sigma \wedge \mathrm{d}y^\sigma \wedge \omega_i + \mathrm{d}\eta - p_3 \,\mathrm{d}\eta. \tag{4.53}$$

Computing $i_{\partial/\partial p^i_\sigma} \hat{\alpha}$, and taking into account that, by assumption, η does not depend upon the p's, we immediately get

$$i_{\partial/\partial p^i_\sigma} \hat{\alpha} = -\frac{\partial H}{\partial p^i_\sigma}\omega_0 + \mathrm{d}y^\sigma \wedge \omega_i = \left(y^\sigma_i - \frac{\partial H}{\partial p^i_\sigma} \right) \omega_0 + \omega^\sigma \wedge \omega_i.$$

However, by (4.52) and (4.48),

$$\begin{aligned} \frac{\partial H}{\partial p^i_\sigma} &= -\frac{\partial L}{\partial p^i_\sigma} + y^\sigma_i + p^k_\nu \frac{\partial y^\nu_k}{\partial p^i_\sigma} + 4g^{pk}_{\rho\nu} y^\rho_p \frac{\partial y^\nu_k}{\partial p^i_\sigma} \\ &= y^\sigma_i + \left(p^k_\nu + 4g^{kp}_{\nu\rho} y^\rho_p - \frac{\partial L}{\partial y^\nu_k} \right) \frac{\partial y^\nu_k}{\partial p^i_\sigma} = y^\sigma_i. \end{aligned} \tag{4.54}$$

Hence, $i_{\partial/\partial p^i_\sigma} \hat{\alpha} = \omega^\sigma \wedge \omega_i$, as desired. □

Note that by (4.54), the matrix

$$\left(\frac{\partial^2 H}{\partial p^i_\sigma \partial p^k_\nu} \right) \tag{4.55}$$

is regular and inverse to the matrix (4.49).

Definition 4.8. We call the functions H (4.52) and p^i_σ (4.48) the *Hamiltonian* and *momenta* of α, respectively.

Remark 4.6. Formulas (4.48) and (4.52) for a Hamiltonian and momenta can be equivalently expressed in terms of a *second-order Lagrangian* equivalent with L. Using notations of Corollary 4.5 and its proof, we obtain

$$p^j_\sigma = \frac{\partial L_0}{\partial y^\sigma_j} - \frac{\partial f^j}{\partial y^\sigma} - \frac{\partial h^{jk}_\sigma}{\partial x^k} - \left(\frac{\partial h^{jk}_\sigma}{\partial y^\nu} + 4g^{jk}_{\sigma\nu}\right) y^\nu_k,$$
$$H = -L_0 + \frac{\partial L_0}{\partial y^\sigma_j} y^\sigma_j + \frac{\partial f^j}{\partial x^j} - \frac{\partial h^{jk}_\sigma}{\partial x^k} y^\sigma_j - \left(\frac{\partial h^{jk}_\sigma}{\partial y^\nu} - 2g^{jk}_{\sigma\nu}\right) y^\sigma_j y^\nu_k. \tag{4.56}$$

If, in particular, L is of the form (3.37), we have $f^j = h^{ij}_\sigma y^\sigma_i$, and the above formulas take the form

$$p^j_\sigma = \frac{\partial L_0}{\partial y^\sigma_j} - \frac{\partial h^{jk}_\sigma}{\partial x^k} - \left(\frac{\partial h^{jk}_\sigma}{\partial y^\nu} + \frac{\partial h^{kj}_\nu}{\partial y^\sigma} + 4g^{jk}_{\sigma\nu}\right) y^\nu_k,$$
$$H = -L_0 + \frac{\partial L_0}{\partial y^\sigma_j} y^\sigma_j - \left(\frac{\partial h^{jk}_\sigma}{\partial y^\nu} + 2g^{jk}_{\sigma\nu}\right) y^\sigma_j y^\nu_k, \tag{4.57}$$

where we recognize (3.40) and (3.42) for $g^{ij}_{\sigma\nu} = 0$ (i.e., for $\rho = \theta_\lambda$).

In Legendre coordinates, we get $\mathcal{D}_{\hat\alpha}$ spanned by the following n-forms:

$$i_{\partial/\partial p^i_\sigma}\hat\alpha = -\frac{\partial H}{\partial p^i_\sigma}\omega_0 + \mathrm{d}y^\sigma \wedge \omega_i,$$
$$i_{\partial/\partial y^\sigma}\hat\alpha = \left(\frac{\partial H}{\partial y^\sigma} - 4\frac{\partial g^{ij}_{\sigma\nu}}{\partial x^i} y^\nu_j - 2\left(\frac{\partial g^{ij}_{\nu\rho}}{\partial y^\sigma} + \frac{\partial g^{ij}_{\sigma\nu}}{\partial y^\rho} + \frac{\partial g^{ij}_{\rho\sigma}}{\partial y^\nu}\right) y^\nu_i y^\rho_j\right)\omega_0 + \mathrm{d}p^i_\sigma \wedge \omega_i. \tag{4.58}$$

If $\mathrm{d}\eta = 0$, the generators of $\mathcal{D}_{\hat\alpha}$ become

$$-\frac{\partial H}{\partial p^i_\sigma}\omega_0 + \mathrm{d}y^\sigma \wedge \omega_i, \qquad \frac{\partial H}{\partial y^\sigma}\omega_0 + \mathrm{d}p^i_\sigma \wedge \omega_i. \tag{4.59}$$

Hamilton equations in Legendre coordinates thus read as follows.

Theorem 4.5. *A Dedecker's section $\delta : U \to W$ is a Dedecker–Hamilton extremal of α* (4.46) and (4.47) *iff it satisfies the equations*

$$\frac{\partial y^\sigma}{\partial x^i} = \frac{\partial H}{\partial p^i_\sigma},$$
$$\frac{\partial p^i_\sigma}{\partial x^i} = -\frac{\partial H}{\partial y^\sigma} + 4\frac{\partial g^{ij}_{\sigma\nu}}{\partial x^i}\frac{\partial H}{\partial p^j_\nu} + 2\left(\frac{\partial g^{ij}_{\nu\rho}}{\partial y^\sigma} + \frac{\partial g^{ij}_{\sigma\nu}}{\partial y^\rho} + \frac{\partial g^{ij}_{\rho\sigma}}{\partial y^\nu}\right)\frac{\partial H}{\partial p^i_\nu}\frac{\partial H}{\partial p^j_\rho}. \tag{4.60}$$

If $\mathrm{d}\eta = 0$, (4.60) *read*

$$\frac{\partial y^\sigma}{\partial x^i} = \frac{\partial H}{\partial p^i_\sigma}, \qquad \frac{\partial p^i_\sigma}{\partial x^i} = -\frac{\partial H}{\partial y^\sigma}. \tag{4.61}$$

In view of Theorem 4.4, we can state the following definition.

Definition 4.9. Let α be a regular Hamiltonian system on J^1Y associated with a first-order Lagrangian system, let $W \subset J^1Y$ be an open set. We say that α *admits Legendre transformation on* W if $\alpha = \mathrm{d}\rho$ for a Lepagean n-form ρ on W such that $p_2\rho = p_2\beta$ for a $\pi_{1,0}$-projectable form β on W (i.e., locally, $p_2\rho = g^{ij}_{\sigma\nu}\omega^\sigma \wedge \omega^\nu \wedge \omega_{ij}$, where $g^{ij}_{\sigma\nu}$ are functions on $\pi_{1,0}(W)$).

5.1. Strong regularity

Let us consider Hamiltonian systems on J^1Y, associated with first-order Lagrangian systems. We have seen that in this case regularity of α guarantees *bijective correspondence between extremals of E and the* $\pi_{2,1}$*-projections of Dedecker–Hamilton extremals of* α, i.e., those solutions of the Hamilton equations (4.7) which annihilate all at least 2-contact forms. Now, we shall study under what conditions there arises a *bijective correspondence between extremals and Hamilton extremals* (integral sections of the Hamiltonian ideal $\mathcal{D}_\alpha$ related with α).

Definition 4.10. A *Hamiltonian system* α will be called *strongly regular* if Hamilton extremals of α are in bijective correspondence with extremals of $E = p_1\alpha$.

A *Lagrangian system* will be called *strongly regular* if it has a strongly regular associated Hamiltonian system.

Proposition 4.7. *Every strongly regular Hamiltonian system is regular.*

Proof. Indeed, since every Hamilton extremal of α is of the form $\delta = J^1\gamma$, every Dedecker–Hamilton extremal $\hat{\delta}$ of α satisfies $\pi_{2,1} \circ \hat{\delta} = J^1\gamma$. This means that for all σ and i, $\hat{\delta}^*\omega^\sigma \wedge \omega_i = 0$, i.e., all the canonical 1-contact n-forms belong to $\mathcal{D}_{\hat{\alpha}}$. Consequently, $\mathcal{D}_{\hat{\alpha}}$ is locally generated by the forms (4.32). Since the forms η^p_ρ in (4.26) must be linear combinations of the canonical 1-contact n-forms, the conditions (4.30) and (4.31) of Theorem 4.3 are satisfied, proving that α is regular. □

From Theorem 4.4, we obtain the following fundamental result which gives us another geometric meaning of Legendre transformations. Roughly speaking, it says that *Hamiltonian systems which admit Legendre transformations are* either *strongly regular*, or *"almost strongly regular"* (a strongly regular Hamiltonian systems appears by modifying the term μ_3 which, however, does not enter into the corresponding Euler–Lagrange equations). Thus, one gets a *characterization of Lepagean forms for which Hamilton and Euler–Lagrange equations are equivalent.*

Theorem 4.6. *Let α be a Hamiltonian system admitting Legendre transformation on $W \subset J^1Y$. If (in the notations of Theorem 4.4) the at least 3-contact part μ_3 of ρ is $\pi_{2,0}$-projectable then $\alpha = \mathrm{d}\rho$ is strongly regular on W.*

Proof. Consider Hamilton equations $\delta^* i_\xi \, \mathrm{d}\rho = 0$ in the Legendre coordinates. The coordinate expression is obtained from ρ (4.50), by contracting by the vector fields $\partial/\partial y^\sigma$ and $\partial/\partial p^i_\sigma$. Since, by assumption, μ_3 is $\pi_{2,0}$-projectable, $\mathrm{d}\mu_3$ does not depend upon momenta, hence, its contraction by $\partial/\partial p^i_\sigma$ is 0. Thus, the corresponding set of *Hamilton* equations takes the form

$$\frac{\partial y^\sigma}{\partial x^i} = \frac{\partial H}{\partial p^i_\sigma}$$

(i.e., the same as the corresponding set of Dedecker–Hamilton equations). However, every section δ of π_1 satisfying these equations is of the form $\delta = J^1\gamma$ for a section γ of π. In other words, every Hamilton extremal of $\mathrm{d}\rho$ is *holonomic*, which means that Hamilton equations of $d\rho$ are equivalent with the Euler–Lagrange equations. □

Corollary 4.8.

(1) *Let α be a first-order Hamiltonian system. Assume that there is an open covering $\{W_\iota\}$ of J^1Y such that α satisfies the conditions of Theorem 4.6 on each W_ι. Then, α is strongly regular, i.e., Hamilton and Euler–Lagrange equations of α are equivalent.*
(2) *Under assumptions of Theorem 4.6, (4.60) (respectively, (4.61)) are equations for Hamilton extremals of α.*
(3) *Let ρ be a Lepagean n-form on J^1Y, i.e. $\pi^*_{2,1}\rho = \theta_\lambda + \mu + \mathrm{d}\nu$, where μ is at least 2-contact, and ν is an $(n-1)$-form. Assume that μ is $\pi_{2,0}$-projectable. If ρ is regular then Hamilton and Euler–Lagrange equations of ρ are equivalent.*

Note that regularity in (3) above means that $\mathrm{d}\rho$ satisfies regularity condition (4.31), which for first- and second-order Lagrangians becomes Dedecker's regularity condition (3.35) (cf. (4.33)) and the generalized Krupka–Štěpánková condition (4.34), respectively.

Remark 4.7 (On variational problems with fibre dimension $m = 1$). Let us consider first-order Lagrangian systems on a fibred manifold with $m = 1$ (and $n = \dim X$ arbitrary). This case is quite specific, since the at least 2-contact part of any $\pi_{1,0}$-horizontal n-form on J^1Y identically vanishes (indeed, it contains wedge products of at least two copies of the contact forms $\omega = \mathrm{d}y - y_k\,\mathrm{d}x^k$). Taking into account Remark 4.4, we can see that (similarly as in mechanics) every first-order Lagrangian has a *unique* first-order $\pi_{1,0}$-*horizontal* Lepagean equivalent, $\rho = \theta_\lambda$. Now, we immediately obtain the following proposition.

Proposition 4.8. *Let $\pi : Y \to X$ be a fibred manifold,* $\dim X = n \geq 2$, $m = \dim Y - n = 1$. *Let E be a dynamical form on J^2Y. Assume that E is locally variational and defines a first-order Lagrangian system. Then there exists a unique Lepagean $(n+1)$-form α on J^1Y such that (locally) $\alpha = \mathrm{d}\rho$, where ρ is $\pi_{1,0}$-horizontal, and $p_1\alpha = E$.*

In terms of (4.20) and with the notations $\omega_j = \mathrm{d}y_j - y_{jk}\,\mathrm{d}x^k$, $\Omega_j = i_{\partial/\partial x^j}\omega_0$, we have $\phi = 0$, and

$$\alpha = \hat{\alpha} = \epsilon\omega \wedge \omega_0 + \frac{\partial \epsilon}{\partial y_{ij}}\omega \wedge \omega_j \wedge \Omega_i = \mathrm{d}\theta_\lambda,$$

where λ is any (local) first-order Lagrangian for E.

Note that for Hamiltonian systems described by Proposition 4.8, the concepts of regularity and strong regularity coincide. Moreover, one has a *unique regularity condition* which depends only on the corresponding Euler–Lagrange expression,

$$\det\left(\frac{\partial \epsilon}{\partial y_{ij}}\right) \neq 0. \tag{4.62}$$

Similarly as in Corollary 4.5, it can be expressed by means of particular Lagrangians (possibly of different orders). Moreover, if L' and L are equivalent first-order Lagrangians, i.e., if $L' = L + d_i f^i$, where $\partial f^i/\partial y_j + \partial f^j/\partial y_i = 0$, then (since the latter condition gives $\partial^2 f^i/\partial y_j \partial y_k = 0$),

$$\frac{\partial^2 L'}{\partial y_i \partial y_j} = \frac{\partial^2 L}{\partial y_i \partial y_j} = \frac{\partial \epsilon}{\partial y_{ij}}. \tag{4.63}$$

Hence, if expressed by means of *any* first-order Lagrangian, the regularity condition (4.62) takes the standard form (3.6), showing that if a first-order Lagrangian satisfies the regularity condition (3.6), then *every* equivalent Lagrangian of the same order satisfies this condition as well. Accordingly, Legendre transformation (momenta, Hamiltonian) are determined from *any* first-order Lagrangian of E by standard formulas. Note that none of these properties is saved if $m > 1$. Comparing these results with corresponding properties of mechanical Lagrangian systems (i.e., $\dim X = 1$, m arbitrary) (cf. [46]), we can see that the case $m = 1$ and $n > 1$ is more similar to mechanics than to field theory.

A typical example of a (regular) Lagrangian system of this kind is the familiar *scalar field* (i.e., the *Klein–Gordon equation*).

Remark 4.8 (On Hamilton p_2-equations). Let $W \subset J^1Y$ be an open set, consider a Lepagean n-form ρ on W, such that

(1) ρ is *at most* 2-*contact*,
(2) $p_2\rho$ is of the form $p_2\rho = g^{ij}_{\sigma\nu}\omega^\sigma \wedge \omega^\nu \wedge \omega_{ij}$, where $g^{ij}_{\sigma\nu} = -g^{ji}_{\sigma\nu} = -g^{ij}_{\nu\sigma}$, and the $g^{ij}_{\sigma\nu}$ are functions of (x^k, y^ρ).

Recall that by Definition 4.4, the corresponding Hamilton equations, $\delta^* i_\xi\,\mathrm{d}\rho = 0$, are called *Hamilton* p_2-*equations*.

Denote $\lambda = h\rho = L\omega_0$. Applying the results obtained so far to the above case of Lepagean forms, we immediately recover the following assertions, recently proved in [50] in connection with the study of Hamilton p_2-equations for first-order Lagrangians.

Assume that ρ satisfies the regularity condition (4.33), where $\Lambda^{ij}_{\sigma\nu} = 4g^{ij}_{\sigma\nu}$. Then

(1) *The Hamiltonian system $\alpha = \mathrm{d}\rho$ is strongly regular.*
(2) *Hamilton equations of $\mathrm{d}\rho$ and Euler–Lagrange equations of $E_\lambda = p_1\,\mathrm{d}\rho$ are equivalent.*

(3) *The Hamiltonian system* $\mathrm{d}\rho$ *admits Legendre transformation on W. In Legendre coordinates, Hamilton equations take the form* (4.60) *with momenta and Hamiltonian defined by* (4.48) and (4.52), *respectively. If, moreover, the n-form* $g^{ij}_{\sigma\nu}\,\mathrm{d}y^\sigma\wedge\mathrm{d}y^\nu\wedge\omega_{ij}$ *is closed, Hamilton equations take the "standard" form* (4.61).

From the point of view of the general Hamiltonian theory, Hamilton p_2-equations can be viewed as a "first correction" to Hamilton–De Donder equations (adding to the Poincaré–Cartan form θ_λ a "free" 2-contact term); in this sense, "higher corrections" are represented by adding to θ_λ higher contact terms, starting from a 2-contact one. However, in view of the above results, we can see that *within the general Hamilton theory, Hamilton* p_2*-equations play a distinguished role*. Indeed, this class of Hamiltonian systems is sufficiently general on one hand, and as simple as possible on the other hand for obtaining *equivalent Hamiltonian counterparts of a given variational problem*, and constructing *coordinate transformations* (*Legendre transformations*) *canonically adapted to the Hamiltonian differential system*.

For more details on Hamilton p_2-equations, and their applications in the calculus of variations and in physics, we refer to [49,50,58].

Remark 4.9 (Regularizable Lagrangians). We shall finish this paper by mentioning some applications of general Hamilton equations in the theory of Lagrangian systems, as discussed (from a different point of view) in [50] (cf. also [5]). Given a *first-order Lagrangian system*, let us study the existence of related *strongly regular* Hamiltonian systems. Recall that by Definition 4.2 and Remark 4.1 a first-order Lagrangian system is an equivalence class, *representable by a family of local equivalent Lagrangians on* J^1Y. In every fibred chart, the Euler–Lagrange expressions E_σ of a first-order Lagrangian system are functions *affine in the "second derivatives"* (in particular, the $\partial E_\sigma/\partial y^\nu_{ij}$ may identically equal 0). Note that in the latter case, the existence of first-order Lagrangians *affine* in the "first derivatives", y^ρ_p, is *equivalent* with the requirement that the E_σ *be functions affine in the first derivatives*, as well.

Taking into account *regularity conditions* for $[\alpha]$ (Theorem 4.3 and its corollaries), we can see immediately that *on fibred manifolds with the fibre dimension m at least* 2 (and, of course, $n = \dim X \geq 2$), to *every* Lagrangian system one can find local associated Hamiltonian systems which are *regular*. Indeed, let $\det(\partial^2 L/\partial y^\sigma_i\partial y^\nu_j) = 0$ at a point $x \in J^1Y$. Since $m > 2$, one can find functions $\Lambda^{ij}_{\sigma\nu}$, antisymmetric in $(\sigma\nu)$ and (ij), defined in a neighbourhood of x and such that at x the condition (4.33) is satisfied. However, since the determinant is a continuous function, the corresponding matrix must be nondegenerate in a neighbourhood of x. On the other hand, the question on the existence of a Hamiltonian system *equivalent* with a given Lagrangian system is less trivial, since, moreover, one needs the functions $\Lambda^{ij}_{\sigma\nu}$ be independent of the y^ρ_p's.

A first-order *Lagrangian system* $[\alpha]$ is called *locally regularizable* if there exists an open covering $\{W_\iota\}$ of J^1Y, and for every ι, a *strongly regular* Hamiltonian system α_{W_ι} defined on W_ι and belonging to the class $[\alpha]$ on W_ι. A *Lagrangian system* $[\alpha]$ is called *regularizable* if there exists a *strongly regular* associated Hamiltonian system. Accordingly, a *Lagrangian* λ is called *locally regularizable* (respectively, *regularizable*) if the corresponding Lagrangian system is locally regularizable (respectively, regularizable) [50]. A (local) Lepagean n-form

ρ such that $E = p_1\, \mathrm{d}\rho = [\alpha]$ and $\mathrm{d}\rho$ is strongly regular is called a (*local*) *regularization* for E.

Note that regularizability is a property of the *class* of equivalent Lagrangians. The geometric content of regularization consists in transferring the problem of finding extremals (which, as sections passing in Y, have no direct geometric interpretation by means of a differential system on Y) to the problem of finding integral sections of a differential ideal generated by $\mathcal{D}_\alpha$.

Let us consider Lagrangians of the form $L = a + b^i_\sigma y^\sigma_i + c^{ij}_{\sigma\nu} y^\sigma_i y^\nu_j$, where a, b^i_σ, $c^{ij}_{\sigma\nu}$ are functions defined on an open subset of Y. By (3.6), such a Lagrangian is regular if $\det(c^{ij}_{\sigma\nu}) \neq 0$. Thus, in this sense, every affine Lagrangian is degenerate, and the same holds for many quadratic Lagrangians (among others the important *electromagnetic field Lagrangian*). Contrary to that one can see immediately that for quadratic (in particular, affine) Lagrangians in the variables y^ρ_p, similar arguments as above lead to the conclusion that *if the fibre dimension* $m \geq 2$ *then every quadratic* (*respectively, affine*) *Lagrangian is locally regularizable*. Consequently, *for* $m \geq 2$ *every local Lagrangian on* $J^r Y$, $r \geq 1$, *which is equivalent with a quadratic* (*respectively, affine*) *first-order Lagrangian, is locally regularizable and admits Legendre transformation*.

Evidently, for *affine Lagrangians* (i.e., *affine in the* y^ρ_p*'s Euler–Lagrange forms*) a corresponding strongly regular Hamiltonian system takes the form $\alpha = \mathrm{d}\rho$ with $\rho = \theta_\lambda + g^{ij}_{\sigma\nu}\omega^\sigma \wedge \omega^\nu \wedge \omega_{ij}$, where $(g^{ij}_{\sigma\nu})$ is a regular matrix defined on an open subset W of Y and such that $g^{ij}_{\sigma\nu} = -g^{ji}_{\sigma\nu} = g^{ji}_{\nu\sigma}$; the principal part of α reads

$$\hat{\alpha} = E + p_2\, \mathrm{d}\rho = E + \frac{1}{2}\left(\frac{\partial E_\sigma}{\partial y^\nu_i} + d_j g^{ij}_{\sigma\nu}\right)\omega^\sigma \wedge \omega^\nu \wedge \omega_i + 4 g^{ij}_{\sigma\nu}\omega^\sigma \wedge \omega^\nu_j \wedge \omega_i \tag{4.64}$$

(cf. also (4.34), (4.10) and (4.11)). Contrary to formulas which appear in the Hamilton–De Donder theory, momenta (4.48) are *independent* functions on $\pi^{-1}_{1,0}(W) \subset J^1 Y$, affine in the y^ν_j's, and Hamiltonian (4.52) (in Legendre coordinates) is a *polynomial of degree* 2 in momenta. A typical example of a physical field of this kind is the *Dirac field* (see [50] for details).

Moreover, taking into account the Krupka form (4.36), we can see that *if* λ *on* $J^1 Y$ *is quadratic and satisfies the condition* (4.37) *then*

$$\rho^{\mathcal{K}}_\lambda = \theta_\lambda + \frac{1}{4}\frac{\partial^2 L}{\partial y^\sigma_j \partial y^\nu_k}\omega^\sigma \wedge \omega^\nu \wedge \omega_{jk}$$

is a (*global*) *regularization* of the corresponding Lagrangian system.

Finally, the results presented so far give us conditions when *second-order Lagrangians affine in the second derivatives*, and *giving rise to first-order Lagrangian systems* are regularizable (cf. Corollary 4.5, Theorem 4.4 and Remark 4.6). In view of these results one immediately obtains that the *Einstein–Hilbert Lagrangian* of the general relativity theory (“pure gravity”) is regularizable, and its most simple regularization is exactly the one obtained by Krupka and Štěpánková in [43] (cf. also Hořava [28]).

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